Fermionic Symmetries

spatially internal symmetry group of system with fermion

want: mathematical defn

Physics: Time-reversal symmetry $T^2 = (-1)^F$ fermion number operator

Lie group

"anti-unitary" $\mathbb{Z}_4$

Def A fermionic group is a

1. cpt Lie G
2. cts morphism $|.| : G \to \mathbb{Z}_2$ additive   odd means time-reversing
3. Central $c \in G$ s.t. $c^2 = 1$, $|c| = 0$

$\Rightarrow$ subgroup $\mathbb{Z}_2^f$ "$c = (-1)^F$"

Ex $c \in \mathrm{Pin}_d^+ \longrightarrow \mathbb{Z}_2$ labels 2 components

$$\begin{array}{c} \mathrm{Pin}_d^+ \\ \downarrow \\ O_d \end{array}$$

Def K_1, K_2 ferm gps   fermionic tensor product

$$K_1 \otimes K_2 := \frac{K_1 \times K_2}{\mathbb{Z}_2^F}$$

$H \xrightarrow{\theta} \mathbb{Z}_2 \overset{gr}{\curvearrowright} G$

graded tensor product over $\mathbb{Z}_2^F$

$$(K_1 \otimes K_2)(K_1' \otimes K_2') = c^{|K_2||K_1'|} K_1 K_1' \otimes K_2 K_2'$$

$\mathbb{Z}/_2 \times \mathbb{Z}/_2$ -graded $\xrightarrow{+} \mathbb{Z}/_2$ diagonal grading

$$(K_1 \otimes K_2)_{ev} = (K_1)_{ev} \otimes (K_2)_{ev} \sqcup (K_1)_{odd} \otimes (K_2)_{odd}$$

Def Given internal G,

time-reversing symmetries are internal

the <u>spacetime structure group</u> ← for spacetime

in spacetime dim d

Euclidean + fermionic version of Lorentz group

is $H_d := \left(G \otimes \mathrm{Pin}^+(d) \right)_{ev}$

If $G \to O_d$ homomorphism, $\exists$ notion of

G-structure on k-mfds $k < d$ $\Rightarrow$ G-TFTs

Ex. • $c = 1$ & $1.1 = 0$ $\Rightarrow$ $H_d = K \times SO_d$ $\Rightarrow$ orientation + K-principal bundle

• $K = \mathbb{Z}_2^F$ $\Rightarrow$ $H_d = \mathrm{Spin}_d$ $\Rightarrow$ spin structure

to get unoriented need TRS

• $d = 1$ $\Rightarrow$ $H_1 = K^{op}$ $\qquad K_1 * K_2 = (-1)^{|K_1||K_2|} K_2 K_1$

f.e. $K = \mathrm{Pin}_1^- = \mathbb{Z}/4 \Rightarrow K^{op} = \mathrm{Pin}_1^+ = \mathbb{Z}/2 \times \mathbb{Z}/2^f$

Wick rotation exchanges $T^2 = 1$ & $T^2 = (-1)^F$

TFT with fermionic symmetry

Fully extended $\Rightarrow$ cobordism hypothesis   computational tool

interesting braiding
(fermions)
(universal target)

$d=1 \quad Z : \mathrm{Bord}^{H}_{1,0} \to s\mathrm{Vect}_{\mathbb{C}}$

$\bullet^{+} \longmapsto V$ f.d.

$\bullet^{-} \longmapsto V^{*}$

$\overset{+}{\bullet}\!\!\underset{h}{-\!\!\!+\!\!\!-}\!\!\overset{-}{\bullet} \longmapsto \left(V \xleftarrow{F_h} V^{*} \right)$

right to left   s.t. ... rep like

Ex $K = O_1 \xrightarrow{1.1 = id} \mathbb{Z}_2$, $c = 1$ $\Rightarrow$ $H_1 = O_1$ In general O_d

bosonic TRS

$V \in s\mathrm{Vect}^{f.d.}_{\mathbb{C}}$ & $(\cdot,\cdot) : V \times V \to \mathbb{C}$ nondeg bil form
graded symmetric

Physical
Problems : 1. expected $V = V_0$ $\Rightarrow$ spin-statistics connection

2. ——— V Hilbert space $\Rightarrow$ unitarity

& T anti-unitary

We discuss
V hermitian

Actions on categories & their fixed points

Cobordism hypothesis: $\mathrm{TFT}_1^H \simeq \left(s\mathrm{Vect}_{\mathbb{C}}^{f.d.}\right)^H$ ← fixed points

↖ f.d. vspaces with invertible linear maps

Def If $C \in$ 2-Cat, an action of G on C is a group homomorphism $G \xrightarrow{\rho} \mathrm{Aut}\, C$, ∘

Vector spaces

2-group

finite group

monoidal 2-functor

E_1-set map

Its fixed pts are

$$\begin{tikzcd} & & & C \\ \text{2-Cat} \ni C^G \arrow[rrru, bend left] \arrow[rrrd, bend right] \arrow[r] & C \arrow[rru, "\rho(gh)"] \arrow[rrd, "\rho(h)"'] \arrow[rr, phantom, "\Downarrow \rho_{g,h}"] & & \\ & & & C \arrow[uu, "\rho(g)"'] \end{tikzcd}$$

$\rho(g)x \overset{F_g}{\simeq} x$

Ex $G \curvearrowright \mathcal{C}$ trivial ⟹ a fix pt is action on some $X \in \mathcal{C}$

Ex $O(1) \curvearrowright s\mathrm{Vect}_{\mathbb{C}}^{f.d.}$

$$\rho(-)V = V^*$$
$$\rho(-)f = f^{*-1}$$
$$V^{**} \underset{\rho_{g,h}}{\simeq} V$$

$H \to O(1) \curvearrowright s\mathrm{Vect}_{\mathbb{C}}^{f.d.}$

f.d. semisimple

Cob. hyp in d=2: $TFT_2^H \simeq \left(sAlg_{\mathbb{C}}^{fd}\right)^H$

inv f.d. bimodules

inv intertwiners

$O(2)$

extended (C.S.P. has def'n of extended bordism 2-cat)

target Alg

Th (Davidovich) (Müller, S.)

K finite ~~bosonic~~ Fermionic w/o TRS

$$ {}_{SS}TFT_2^H = \left\{ \bigoplus_{k\in K} A_k : \begin{array}{l} \text{strongly } K\text{-graded } sAlg \,+ \\ \text{ungraded sym Frob str} \\ \text{s.t. } \langle a_{k_1}, a_{k_2}\rangle = 0 \\ \text{unless } k_1 = k_2^{-1} \\ \& \ A_c = A(-1)^F \end{array} \right\} $$

spin statistics

Spin-statistics

<u>spin</u> a particle $v \in V$ has spin $s \in \mathbb{Z} + \frac{1}{2}$ if if
half-integer
is only a $Spin_d$-rep, not a SO_d-rep
Lorentz transform

<u>statistics</u> braiding in sVect
i.e. $\mathcal{X}$ 1 particle Hilb of $\mathbb{C}$-fermions, free theories

$\Rightarrow$ Fock space $\Lambda^\bullet \mathcal{X} = \Lambda^{ev}\mathcal{X} \oplus \Lambda^{odd}\mathcal{H}$
many body
$(-1)^{\#\text{particles}} = (-1)^F$

for general $V \in sVect_{\mathbb{C}}$ denote grading $(-1)^F$
interacting

<u>spin-statistics</u>: $(-1)^{2s} = (-1)^F$

<u>Def</u> A nonextended H-TFT Z satisfies <u>spin-statistics</u> if "$c \mapsto (-1)^F$"
$(-1)^{2s}$ before

$\forall\ Y^{d-1}$ H_d-mfd $\Rightarrow$ H_d-diffeo $c_Y : Y \to Y$
c central

$\Rightarrow$ bordism $Y \xrightarrow{c_Y} Y$

$Z(c_Y) = (-1)^F_{Z(Y)}$

<u>Observation</u> spin-statistics $\Leftrightarrow$ $B\mathbb{Z}_2$-equivariance
2-group

$B\mathbb{Z}_2 \curvearrowright sVect$ $\quad V \mapsto (-1)^F_V$

$B\mathbb{Z}_2 \curvearrowright \mathrm{Bord}_{d,d-1}^{H} \quad Y \mapsto C_Y$

Z intertwines these

<u>once extended</u>:

$B\mathbb{Z}_2 \curvearrowright s\mathrm{Alg}_{\mathbb{C}} \qquad A \mapsto A_{(-1)^F} \qquad M \underset{A}{\otimes} A_{(-1)^F} \cong B_{(-1)^F} \otimes_B M$

<u>Def</u> A <u>spin-statistics connection</u> on Z is $B\mathbb{Z}_2$-equivariance data

<u>Unitarity</u> (reflection structure)

$s\mathrm{Vect}_{\mathbb{C}}^{f.d.} \circlearrowleft \mathbb{Z}_2 \qquad V \mapsto \overline{V}^*$ $\mathbb{C}$-conjugate vector space

$\mathbb{Z}_2$-fix pt is a hermitian pairing

$h : \overline{V}^* \to V$ s.t.

$\langle v, w \rangle$

$$\begin{tikzcd} \overline{V}^* \arrow[r, "\overline{h}^*"] \arrow[rr, bend right, "h"'] & \overline{\overline{V}^*}^* \arrow[r, "\cong", "C_\alpha"'] & V \end{tikzcd}$$

$$\langle v, w \rangle = (-1)^{|v||w|} \overline{\langle w, v \rangle}$$

Let Z nonextended

def'n in FH

Idea: given Y^{d-1} $\exists$ $\overline{Y}$ orientation-reversal

& isos $Z(\overline{Y}) \cong Z(Y)^*$

i.p. isos $\overline{Z(\overline{Y})} \cong Z(Y)$ s.t. ....

give a herm pairing on $Z(Y)$

$\mathbb{Z}_2 \curvearrowright sVect_{\mathbb{C}}$ $\quad V \mapsto \overline{V}$ $\quad T \mapsto \overline{T}$

$\mathbb{Z}_2 \curvearrowright Bord^H_{d,d-1}$ $\quad Y \mapsto \overline{Y}$

Def A reflection structure on Z is $\mathbb{Z}_2$-equivariance data.

simplifies time-reversal symmetry

$\mathbb{Z}_2 \times B\mathbb{Z}_2 \curvearrowright sAlg_{\mathbb{C}}$ $\quad A \mapsto \overline{A}$ $\quad M \mapsto \overline{M}$ $\quad \varphi \mapsto \overline{\varphi}$

$(\overline{A})_{(-1)^F} \cong \overline{A_{(-1)^F}}$

Def A spin-statistics reflection H-TFT is a H-TFT with $\mathbb{Z}_2 \times B\mathbb{Z}_2$-equivariance data

how to compute?

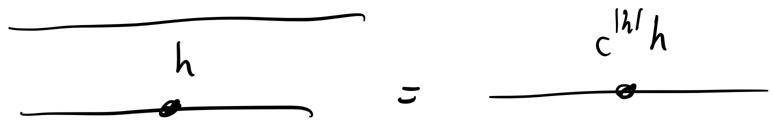
$$rssTFT_2^H = \left(\left(sAlg^{f.d.}\right)^H\right)^{\mathbb{Z}_2 \times B\mathbb{Z}_2}$$

$$\overset{\text{hope?}}{\simeq} \left(\left(sAlg^{f.d.}\right)^{\mathbb{Z}_2 \times B\mathbb{Z}_2}\right)^H \quad \text{not quite}$$

Problem 1 $B\mathbb{Z}_2$ does not act on Bord_2^{fr} $\ni B\mathbb{Z}$

Problem 2 $\mathbb{Z}_2$ acts on Bord_2^{fr}

but



Sol'n (twisted) semidirect product of 2-groups

framed TFTs

$$H \ltimes (\mathbb{Z}_2 \times B\mathbb{Z}_2) \curvearrowright s\mathrm{Alg}_{\mathbb{C}}^{f.d.}$$

$$Rh = c^{|h|} hR$$

$(-1)^F$ acts by $c \in Z(H)$

Lemma $\exists$ s.e.s. of 2-groups (fibration)

$$1 \to \mathbb{Z}_2^R \times \overbrace{(\underbrace{\mathbb{Z}_2^c}_{\text{acts trivially}} \ltimes B\mathbb{Z}_2^f)}^{\text{contractible}} \to H_1 \ltimes (\mathbb{Z}_2^R \times B\mathbb{Z}_2^f) \to H_{1b} \to 1$$

$$\Rightarrow \mathrm{rss\,TFT}_1^H = \left(s\mathrm{Vect}_{\mathbb{C}}^{f.d.}\right)^{H_1 \ltimes (\mathbb{Z}_2^R \times B\mathbb{Z}_2^f)}$$

$$\simeq \left(\underbrace{\left(s\mathrm{Vect}_{\mathbb{C}}^{f.d.}\right)^{\mathbb{Z}_2^R \times (\mathbb{Z}_2^c \ltimes B\mathbb{Z}_2^f)}}_{\text{once \& for all compute this}} \right)^{H_{1b}}$$

Cl: $s\mathrm{Vect}_{\mathbb{C}}^{\mathbb{Z}_2^R \times (\mathbb{Z}_2^c \ltimes B\mathbb{Z}_2^f)} = s\mathrm{Herm}_{\mathbb{C}}$ + unitary maps

$\mathbb{Z}_2^c$: rep of c $\quad$ $B\mathbb{Z}_2^f$: forces $\mathbb{Z}_2^c$-rep to $(-1)^F$

$H_b = K_b$ -action is   For $|h|=1$

$$(V,h) \mapsto (\overline{V}, (-1)^F_{\overline{V}} \circ \overline{h_V})$$

good for positivity

+ interesting ρ_{K_1,K_2}

Th (Müller, S.) rssTFT_1^H = (anti-) unitary reps of K

in the ungraded sense

not H!

Similar in 2d

Prop $\mathrm{sAlg}^{\mathbb{Z}_2^R \times (\mathbb{Z}_2^c \ltimes B\mathbb{Z})} = \mathrm{st\,Frob}$

(ungraded symmetric) Frobius

"stellar"

Ex $*$-algebra $A \Rightarrow A_*$ $(A,\bar{A})$-bim

& $\sigma : \bar{A}_*^{op} \to A_*$ $a \mapsto a^*$

positive $\Leftrightarrow$ C^*

$\bar{A}$ has stellar str $M \circ \bar{A}_{(-1)^F}$

A is $C^* \Rightarrow \bar{A}$ C^*

Th (in progress) rssTFT_2^H

strongly K-graded

$$\bigoplus_{k \in K} A_k \in \mathrm{sVect}_{\mathbb{C}} \quad \mathrm{sAlg}/\mathbb{R}$$

s.t. $i\, a_k = (-1)^{|k|} a_k\, i$

A_1 stellar, A_k stellar bimodules

Hilbert C^*-modules

$\langle \cdot,\cdot \rangle : A_{\bar{k}} \otimes A_k \to A$

multiplication unitary

+ ungr sym Frob str' $\lambda : A_{Cl_1 F} \to A^*$ unitary