Joint work in progress
with Lukas Müller

Two-dimensional Topological Field Theories

&

Spin-Statistics

1. (Non-extended) Topological Field Theories

2. The Spin-Statistics connection

3. Spin-Statistics Connections
on 2d Extended Topological Field Theories

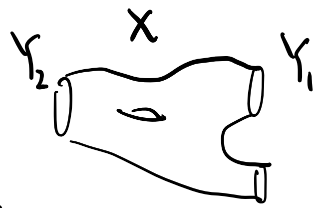
1. The data of a topological field theory

"space": $(d-1)$-dim closed manifold

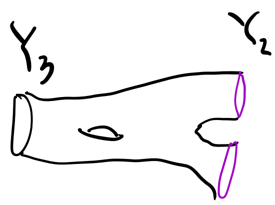
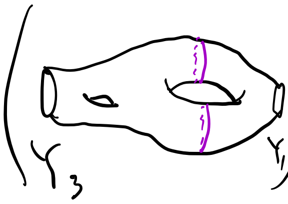
$$F(Y^{d-1}) \in \mathrm{Vect}_{\mathbb{C}} \quad \text{"state space"}$$

$$F(X^{d} : Y_1^{d-1} \rightsquigarrow Y_2^{d-1}) : F(Y_1) \to F(Y_2) \quad \mathbb{C}\text{-linear map}$$

"spacetime": d-dim bordism



These compose under gluing bordisms

$F\Big($  $\Big) \circ F\Big($  $\Big) = F\Big($  $\Big)$

$$F(Y_1 \sqcup Y_2) \cong F(Y_1) \otimes_{\mathbb{C}} F(Y_2)$$

$$F(\emptyset) \cong \mathbb{C}$$

e.g. $\mathrm{Spin}(d)$

<u>Def</u> Let G be a compact Lie group and $\rho: G \to O(d)$ an orthogonal representation.

A (nonextended) <u>d-dimensional G-topological field theory</u> (G-TFT) is a symmetric monoidal functor

$$F : (\mathrm{Bord}^G_{d,d-1}, \sqcup) \to (s\mathrm{Vect}_{\mathbb{C}}, \otimes)$$

manifolds with (G,ρ)-structure

$\mathbb{Z}_2$-graded vector spaces
even linear maps

<u>Example</u> Let $d = 1$ and $G = \mathrm{Spin}(1) = \mathbb{Z}/_2 = \langle c : c^2 = 1 \rangle$

Then a 1d G-TFT is determined by a finite-dimensional super vector space $F(pt)$ and an even representation of G

$F(c) := F\Big($  $\Big) : F(pt) \to F(pt)$

This representation may or may not agree with "fermion parity"

$(-1)^F : F(pt) \to F(pt)\ , \quad v \mapsto (-1)^{|v|} v$

i.e. the super grading of $F(pt)$

<u>Physics sidenote</u>: c is "rotation by 2π",
so if $F(c)\,v = -v$, then v has
"half-integer spin".
Conclusion:

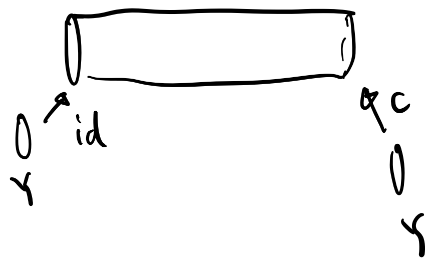
$F(c)$ & $(-1)^F$ agree $\Leftrightarrow$ "a particle has half-integer spin exactly when it has fermionic statistics"

known as the spin-statistics "theorem"

2. $\boxed{\text{Spin \& Statistics}}$

choice of "rotation by 2π"

Def Suppose $c \in G$ is a central element of square 1
s.t. $\rho(c) = 1 \in O(d)$.
A G-TFT satisfies <u>spin-statistics</u> (w.r.t. c) if
for every G-manifold Y^{d-1}
the mapping cylinder

$F(c) := F\Big($ 

$\Big) : F(Y) \to F(Y)$

equals $(-1)^F$.

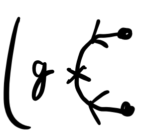
<u>Example</u> $G = SO(1) \overset{1}{=}$ with $c = 1$

A G-TFT is determined by a finite-dimensional $F(pt) \in sVect_{\mathbb{C}}$

F satisfies spin-statistics $\Leftrightarrow$ $F(pt)$ even

<u>Example</u> $G = Pin^-(1) = \mathbb{Z}_4 = \langle g : g^4 = 1 \rangle \twoheadrightarrow O(1)$ with $c \neq 1$.

A G-TFT is determined by a f.d. $\mathbb{Z}_2$-rep $F(c) : F(pt) \to F(pt)$ together with a nondegenerate even bilinear form

$\langle \cdot,\cdot \rangle = F\big($  $\big) : F(pt) \times F(pt) \to \mathbb{C}$

such that $\langle v, w\rangle = (-1)^{|v||w|} \langle F(c) w, v\rangle$

and $\langle v, F(c) w\rangle = \langle F(c) v, w\rangle$.

F satisfies spin-statistics

$\Updownarrow$

$\langle \cdot,\cdot\rangle$ is symmetric (in the ungraded sense)

<u>Observation</u> (Theo Johnson-Freyd)

There are "$B\mathbb{Z}_2$-actions" on

1. $sVect_{\mathbb{C}}$:

$$V \mapsto (-1)^F \in \operatorname{Aut} V$$

2. $Bord_d^G$:

$$Y \mapsto c \in \operatorname{Aut} Y$$

A G-TFT satisfies spin-statistics

$\Updownarrow$

it is $B\mathbb{Z}_2$-equivariant

3. $\boxed{\text{The Two-Dimensional Extended Case}}$

Def A <u>spin-statistics connection</u> on an extended two-dimensional G-TFT

$$F : \mathrm{Bord}^G_{2,1,0} \to s\mathrm{Alg}_{\mathbb{C}}$$

$s\mathrm{Alg}_{\mathbb{C}}$ bicategory: even intertwiners, bimodules, superalgebras

is $B\mathbb{Z}_2$-equivariance data for certain similar $B\mathbb{Z}_2$-actions:

the $B\mathbb{Z}_2$-action on $s\mathrm{Alg}_{\mathbb{C}}$ is

$$A \mapsto A_{(-1)^F} \in (A,A)\text{-bimod}$$

the bimodule induced by the algebra automorphism

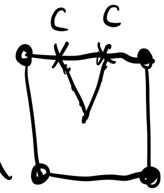
$$a \mapsto (-1)^{|a|} a$$

<u>Explicitly</u> if $A = F(pt) \in s\mathrm{Alg}_{\mathbb{C}}$ & $A_c = F(c) \in \mathrm{Bimod}(A,A)$ this means an (A,A)-bimodule isomorphism

$$A_c \overset{\varphi}{\cong} A_{(-1)^F} \;{}^*$$

such that

$$\begin{tikzcd} A_c \otimes_A A_c \arrow[rr, "\varphi \otimes \varphi"] \arrow[dr] & & A_{(-1)^F} \otimes_A A_{(-1)^F} \arrow[dl] \\ & A & \end{tikzcd}$$

$F\Big($  $\Big) \overset{c^2=1}{\rightsquigarrow}$

* respecting the G-fixed point structure

<u>Example</u> An extended Spin_2-TFT is determined by a f.d. semisimple $A \in \mathrm{sAlg}_{\mathbb{C}}$ together with an (A,A)-bimodule isomorphism

$$A \longrightarrow A^* \otimes_A A^*$$

After identifying $A_{(-1)^F} \cong A_c$, this map will be of the form

$$1 \longmapsto \epsilon \otimes_A \epsilon$$

where $\epsilon : A \to \mathbb{C}$ is an <u>ungraded symmetric</u> Frobenius algebra structure on A.

<u>Theorem</u> (Müller-S.) For $G = \mathrm{Spin}(2) \times K$ with K finite and

$$\rho : \mathrm{Spin}(2) \times K \xrightarrow{pr_1} \mathrm{Spin}(2) \longrightarrow O(2)$$

a spin-statistics G-TFT is given by a strongly K-graded superalgebra

$$\mathcal{A} = \bigoplus_{k \in K} A_k$$

together with an ungraded-symmetric Frobenius structure ϵ on $\mathcal{A}$ such that $\epsilon(a_g a_h) = 0$ if $g \neq h^{-1}$.

<u>More generally</u> Can take $c \in K$ central of square 1.

We have a similar result for

$$G = Spin(2) \times_{\mathbb{Z}_2} K.$$

<u>Outlook</u>: nonorientable case (O_2, Pin_2^-, ...)

<u>Idea</u> Just like spin-statistics makes TFTs "with spin" easier, "reflection structures" make TFTs "with time-reversal" easier