1. Reminder & Motivation for hardcore functional analysis

<u>Recall</u> we study a formal neighborhood of the 'derived' moduli space of solutions to $EL \subseteq$ space of fields

with EL a PDE on manifold M

$\Rightarrow$ sheaf of L_∞-algebras over M

<u>Ex</u> free theory on (M, g) orientable Riemannian

$$L(U) = \left(C^\infty(U)[-1] \xrightarrow{\Delta_g} C^\infty(U)[-2] \right)$$

<u>Def</u> A <u>local L_∞-algebra</u> on M is a $\mathbb{Z}$-graded vector bundle $L \to M$ together with an L_∞-algebra structure on its smooth sections L

$$\ell_n : L^{\otimes n} \to L$$

and the ℓ_n are differential operators

Recall: if $\mathfrak{g}$ L_∞-algebra, had Chevalley-Eilenberg cochains

$$C^*(\mathfrak{g}) = S^\bullet(\mathfrak{g}^*[1]) = \text{"functions on } B\mathfrak{g}\text{"}$$

or $(\mathfrak{g}[1])^*$?

dga with differential Q

Problem: $\mathfrak{g} = \mathfrak{h}(M)$ is huge.

Use a topology? What kind of dual $\mathfrak{g}^*$?

Use algebraic $\otimes$ or something more fancy?

Ex $C^\infty(M_1) \otimes C^\infty(M_2) \not\cong C^\infty(M_1 \times M_2)$

$S^\bullet(\mathfrak{g}^*[1])$ = polynomials on $\mathfrak{h}(M)$.

Want more functions on $\mathfrak{h}(M)$.

Conclusion: to define CE cochains on $\mathfrak{h}$ need:

1. nice topology on $\mathfrak{h}(M)$ $\Rightarrow$ continuous dual
2. nice tensor product

2. Reminder on topological vector spaces

Def a TVS V is a Hausdorff space with cts vector space structure

A locally convex TVS (LCTVS) has its topology generated by a separating collection of seminorms $\{\|\cdot\|_i\}_{i\in I}$

(norms but $\|v\|_i = 0 \not\Rightarrow v = 0$,
However $\forall v \in V$ non zero $\exists \|\cdot\|_i$ s.t. $\|v\|_i \neq 0$)

V is <u>Fréchet</u> if its topology ~~is~~ can be generated by $\{\|\cdot\|_i\}_{i\in I}$
with I countable & V complete.

<u>Example</u> $C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ with

$$\|f\|_{K, i_1, \dots, i_n} := \sup_{x\in K} \left\| \frac{\partial}{\partial x_1^{i_1}} \cdots \frac{\partial}{\partial x_n^{i_n}} f(x) \right\|_{\mathbb{R}^k}$$

$K \subseteq \mathbb{R}^n$ cpt, $i_1, \dots, i_n \geq 0$

is Fréchet.

If $E \to M$ $\mathbb{R}$-vector bundle can now make

$$\mathcal{E}(X) := C^\infty(M; E) \quad \text{Fréchet}.$$

$$\mathcal{D}(X) := \left(C_c^\infty(X) \right)^\vee$$

<u>Example</u> $\bar{\mathcal{E}}(X) := \mathcal{E}(X) \otimes_{C^\infty(X)} \mathcal{D}(X)$ distributions

$$\mathcal{E}^!(X) := C^\infty(X; E^\vee \otimes \mathrm{Dens}_X)$$

bundle of volume densities

Th $\mathrm{Hom}_{cts}(\mathcal{E}(X), \mathbb{R}) \cong \overline{\mathcal{E}}^!_c(X)$ with "strong" topology

$\mathrm{Hom}_{cts}(\mathcal{E}_c(X), \mathbb{R}) \cong \overline{\mathcal{E}}^!(X)$

$\mathcal{E}_c \hookrightarrow \overline{\mathcal{E}}_c \simeq \mathcal{D}_c \leftarrow \Gamma^{-\infty}$ generalized sections

$\overline{\mathcal{E}}_c \hookrightarrow \overline{\mathcal{E}} \simeq \mathcal{D}$

$\mathcal{E}_c \hookrightarrow \mathcal{E} \hookrightarrow \overline{\mathcal{E}}$

Def For V_1, V_2 TVS, the *projective tensor product*

$$V_1 \otimes_\pi V_2$$

is the algebraic tensor with the strongest topology such that

$$V_1 \times V_2 \to V_1 \otimes_\pi V_2$$

is continuous. The *completed projective tensor product*

is the completion $\quad V_1 \overline{\otimes}_\pi V_2 := \overline{V_1 \otimes_\pi V_2}$

Th $C^\infty(M_1) \overline{\otimes}_\pi C^\infty(M_2) \cong C^\infty(M_1 \times M_2)$

We can now happily define *local L_∞-cochains* as

$$\mathcal{O}(\mathfrak{h}(U)[1]) \text{ "}\sim S^\bullet(\mathfrak{g}^*[1])\text{"}$$

$$\shortparallel$$

$$\prod_{n\geq 0} \mathrm{Hom}_{\mathrm{cts}}(\underbrace{\mathfrak{h}(U)[1]\,\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathfrak{h}(U)[1]}_{n}, \mathbb{R})_{S_n}$$

<u>Next problem</u>: homological algebra in Fréchet spaces is hard

<u>Solution</u>: embed in a larger category where homological algebra works O.K: <u>differentiable vector spaces</u>

$\boxed{\text{3. Smooth \& Differentiable vector Spaces}}$

Let Mfd be the site of manifolds without boundary with covers given by surjective local diffeomorphisms

Example if $U_i \subseteq M$ open cover of $M \in \mathrm{Mfd}$, then $\{U_i\}$ is a cover.

Def A *smooth vector space* V is a sheaf of vector spaces on Mfd.

$$V(M) \to \prod_i V(U_i) \rightrightarrows \prod_{i,j} V(U_i \cap U_j)$$

Def *Concrete sheaf*: $V(M) \subseteq \mathrm{Map}_{Set}(M, V(pt))$

Example $V = C^\infty$ does $M \mapsto C^\infty(M)$

Def A *C^∞-module* V is a module over C^∞ in smooth vector spaces, i.e. a smooth vector space together with sheaf maps $V(M) \times V(M) \to V(M)$ etc

$\cdot : C^\infty \times V \to V$, $+ : V \times V \to V$

s.t. ....

Write the value of V at M suggestively as

$C^\infty(M; V)$

__Example__ Let $E \to X$ vector bundle with smooth sections $\mathcal{E}$.

Define the C^∞-module $\mathcal{E}(X)$ as

$$C^\infty(M, \mathcal{E}(X)) := C^\infty(M \times X; \pi_X^* E)$$

__Example__ $\Omega^k(-)$ for fixed k

More generally for V any C^∞-module define

$$\Omega^k(M, V) := \Omega^n(M) \otimes_{C^\infty(M)} C^\infty(M; V)$$

__Def__ A __differentiable vector space__ (DVS) is a C^∞-module V with a "flat connection" ∇: a collection of linear maps

$$\nabla_M : C^\infty(M; V) \longrightarrow \Omega^1(M; V) \text{ s.t.}$$

M a manifold ✓, NOT a vector field

1. $f: M_1 \to M_2$ smooth

$\Downarrow$

Sheaf map

$$\begin{tikzcd} C^\infty(M_1; V) \arrow[r, "\nabla_{M_1}"] \arrow[d, "f^*"] & \Omega^1(M_1; V) \arrow[d, "f^*"] \\ C^\infty(M_2; V) \arrow[r, "\nabla_{M_2}"] & \Omega^1(M_2; V) \end{tikzcd}$$

2. $\nabla_M(f \cdot s) = df \cdot s + f \cdot \nabla_M s$

Leibniz rule $\qquad f \in C^\infty(M),\ s \in C^\infty(M;V)$

3. $F(\nabla_M) = 0$ $\qquad$ flat

which I guess means that the composition

$$C^\infty(M;V) \xrightarrow{\nabla_M} \Omega^1(M;V) \xrightarrow{d^{\nabla_M}} \Omega^2(M;V)$$

is zero

Write DVS for the category of DVS's

Th DVS is a 'Grothendieck category'

(for category fetishists: DVS is abelian, complete, cocomplete & locally accessible. Filtered colimits of exact sequences are exact, which implies DVS has enough injectives)

Def A differentiable chain complex V is a chain complex in DVS

A map $f: V \to W$ of differentiable chain complexes is a quasi-iso if

$$C^\infty(M; V) \xrightarrow{f_M} C^\infty(M; W)$$

is a quasi-iso $\forall M$

4. How to get a DVS from a LCTVS

Def V l.c.t.v.s.

A map $\gamma: \mathbb{R} \to V$ is differentiable if

$$\lim_{s \to 0} \frac{\gamma(t+s) - \gamma(t)}{s} \in V$$

exists for all $t \in \mathbb{R}$. It is smooth if it is differentiable infinitely often.

Let $U \subseteq \mathbb{R}^n$ open. A map $f: U \to V$ is smooth if $f \circ \gamma : \mathbb{R} \to V$ is smooth for every smooth $\gamma: \mathbb{R} \to U$

Lemma f smooth $\iff$ all partial derivatives exist

Def X smooth manifold.

$f : X \to V$ is <u>smooth</u> if it is smooth in every chart.

$$\text{"}TV\text{"} \\ // \\ Tf : TX \to V \oplus V$$ is defined as follows.

Let $u \in T_x X$ & let $\gamma : \mathbb{R} \to X$ be a curve s.t. $\gamma(0) = x$, $\gamma'(0) = u$.

Define

$$Tf(u) = \left(f(x), \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \right)$$

Th V l.c.t.v.s., $C^\infty(X; V) := \{ \text{smooth } f : X \to V \}$ is a D.V.S. with flat connection

$$C^\infty(X;V) \xrightarrow{\nabla_X} \Omega^1(X;V) = \Omega^1(X) \otimes_{C^\infty(X)} C^\infty(X;V)$$

$$f \longmapsto \left(u \mapsto Tf(u) \right)$$

$$TX \to V$$

This defines a functor LCTVS $\to$ DVS which is faithful & preserves limits.

Example $E \to M$ vector bundle

$\Rightarrow \mathcal{E}(M)$ is now a DVS in two ways

Th They are the same

4. Pro-Vector spaces

Clearly DVS's are not complicated enough, so let's give them more data

Def A <u>differential pro-cochain complex</u> is a differential cochain complex together with a filtration

$$\begin{tikzcd} \cdots \arrow[r, hook] & F_2 W \arrow[r, hook] \arrow[d] & F_1 W \arrow[r, hook] \arrow[d] & W \arrow[d] & \leftarrow \text{morphism} \\ \cdots \arrow[r, hook] & F_2 V \arrow[r, hook] & F_1 V \arrow[r, hook] & F_0 V = V \end{tikzcd}$$

Such that the canonical map $V \to \lim_n \dfrac{V}{F_n V}$ is a quasi-iso. A map $f: V \to W$ of differential pro-cochain complex is a <u>filtered weak equivalence</u> if the associated graded maps

$$\mathrm{Gr}_n f : \frac{F_n V}{F_{n+1} V} \to \frac{F_n W}{F_{n+1} W}$$

are quasi-isos $\forall n$

<u>Idea</u> $V = \mathbb{C}[\![x]\!]$ with $F_n V = \{ \sum_{i=n}^{\infty} a_n x^n \}$ $\mathcal{O}(x^n)$

is a pro-vector space

L $\mathbb{Z}$-graded v.bundle

$\mathfrak{h}(U) = \Gamma(U; L)$

<u>Main example</u> $V = \prod_{k \geq 0} \mathrm{Hom}_{cts}\left((\mathfrak{h}(U)[1])^{\hat{\otimes}_\pi k}, \mathbb{R} \right)_{S_k}$

with $F^n V = \prod_{k \geq n} \mathrm{Hom}_{cts}\left((\mathfrak{h}(U)[1])^{\hat{\otimes}_\pi k}, \mathbb{R} \right)_{S_k}$

ϕ

<u>Def</u> This is the <u>Chevalley-Eilenberg complex</u> of a local L_∞-algebra $\mathfrak{h}$.

Can also define a "local action" of $\mathfrak{h}$ on $\mathcal{E}$ = sections of graded vector bundle E

$\Rightarrow$ $\mathcal{E}$-valued C.E. cochains.