The spin-statistics theorem for TQFTs

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Outline

1. spin & statistics
   1. the statistics of bosons and fermions
   2. the spin of a particle
   3. implementation for TQFTs
2. unitary TQFT
   1. Hilbert spaces & $\dagger$-categories
   2. $\mathbb{Z}/2$-actions & hermitian pairings
   3. unitary vs reflection positive TQFTs
3. the spin-statistics theorem
   1. fermionically $\dagger$-compact categories
   2. the proof
4. outlook (time-permitting)
   1. higher spin statistics theorems
   2. O-unitary TQFTs

Statistics: super vector spaces

Physically:

- State spaces are graded by $(-1)^F$
- Bose statistics $\phi_1\phi_2 = \phi_2\phi_1$
- Fermi statistics $\psi_1\psi_2 = -\psi_2\psi_1$

Mathematically:

- symmetric monoidal category sVect of super vector spaces

Spin and the spin group

- for $d = 3$, irreps of $\mathrm{Spin}(d)$ are given by spins $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$
- integer spin $\iff$ descends to $\mathrm{SO}(d)$
- for arbitrary d, the kernel $\mathrm{Spin}(d) \to \mathrm{SO}(d)$ is generated by an element $(-1)^{2s} \in \mathrm{Spin}(d)$
- if (V, R) is an irrep of $\mathrm{Spin}(d)$, we say $v \in V$ has *half-integer* spin if $R((-1)^{2s})v = -v$
  and *integer spin* if $R((-1)^{2s})v = v$

The spin-statistics theorem

“Theorem”

In a unitary QFT, a particle has half integer spin if it is a fermion and integer spin if it is a boson.
In other words, $(-1)^{2s} = (-1)^F$.

TQFTs

From a state perspective, a d-dimensional QFT assigns

- to $(d-1)$-dimensional manifold Y ('space') a vector space of states $Z(Y)$
- to a d-dimensional bordism $Y_1 \rightsquigarrow Y_2$ a linear map $Z(X) : Z(Y_1) \to Z(Y_2)$ ('time evolution')

When we include fermions and spinors:

- state spaces should be graded by $(-1)^F$
- spin structures on spacetime allow for definitions of spinors in QFT.

Definition

A *fermionic TQFT* is a symmetric monoidal functor

$$Z : (\mathrm{Bord}_d^{\mathrm{Spin}}, \sqcup) \to (\mathrm{sVect}, \otimes).$$

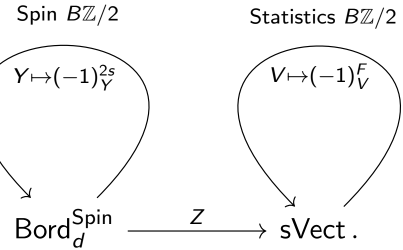
Spin and statistics for TQFTs

- multiplication by $(-1)^{2s} \in \mathrm{Spin}(d)$ gives an involution on a $\mathrm{Spin}(d)$-principal bundle.
- an automorphism $Y \to Y$ of a spin manifold induces a bordism $Y \rightsquigarrow Y$.
- we obtain an involution $(-1)^{2s}_Y$ of every time slice $Y \in \mathrm{Bord}_d^{\mathrm{Spin}}$
- This determines an involution $Z((-1)^{2s}_Y)$ on the state space $Z(Y)$
- the supergrading gives the involution $(-1)^F_{Z(Y)}$

Connecting spin and statistics

Definition

A d-dimensional fermionic TQFT Z has a *spin-statistics connection* if it is $B\mathbb{Z}/2$-equivariant.



Theorem (S.)

Every unitary fermionic TQFT has a spin-statistics connection.

How to define unitary TQFT?

$\dagger$-categories

- State spaces of a unitary QFT have a Hilbert space structure
- Hilbert spaces form a $\dagger$-category: $\dagger : \mathrm{Hilb} \to \mathrm{Hilb}^{\mathrm{op}}$ such that $\dagger^2 = \mathrm{id}_{\mathcal{C}}$ and $\dagger(V) = V$
- Atiyah: the oriented bordism category is a $\dagger$-category by orientation reversal
- a bosonic unitary TQFT is a symmetric monoidal $\dagger$-functor $\mathrm{Bord}_d^{SO} \to \mathrm{Hilb}$

How to build $\dagger$ on $\mathrm{Bord}_d^{\mathrm{Spin}}$?

Just like in Hilb: from $\mathbb{Z}/2$-actions and Hermitian pairings.

Super Hilbert spaces

- complex conjugation defines a $\mathbb{Z}/2$-action on super vector spaces $\overline{(.)}$: sVect $\to$ sVect
- A super Hermitian vector space is a nondegenerate pairing $h : V \otimes \overline{V} \to \mathbb{C}$ such that

$$\begin{array}{ccc} \overline{V \otimes \overline{V}} & \xrightarrow{\bar{h}} & \mathbb{C} \\ \downarrow & & \uparrow h \\ \overline{V} \otimes \overline{\overline{V}} & \longrightarrow & \overline{V} \otimes V \end{array}$$

commutes
- (finite-dimensional) super Hilbert spaces: Hermitian spaces with positivity condition

Build a † from Hermitian pairings

Definition

Let $\overline{(.)} : \mathcal{C} \to \mathcal{C}$ be a $\mathbb{Z}/2$-action on a symmetric monoidal category. The *Hermitian completion of* $\mathcal{C}$ is the category in which objects are pairs (x, h) with $x \in \mathcal{C}$ and $h : \overline{x} \otimes x \to 1$ a nondegenerate Hermitian pairing. Morphisms are simply morphisms in $\mathcal{C}$.

- h makes $\overline{x}$ into a dual of x
- $\Rightarrow \operatorname{Herm}\mathcal{C}$ has a dual functor $(.)^* : \operatorname{Herm}\mathcal{C} \to \operatorname{Herm}\mathcal{C}^{\mathrm{op}}$
- it is a †-category via $\dagger = \overline{(.)}^*$
- if P is a collection of nondegenerate Hermitian pairings on $\mathcal{C}$, let $\mathcal{C}_P$ denote the full subcategory of $\operatorname{Herm}\mathcal{C}$ generated by P

The $\mathbb{Z}/2$-action on the bordism category

The orientation reversal $\mathbb{Z}/2$-action can be generalized to spin manifolds as follows.

Given a Spin_d-bundle P, there is an associated '$\mathbb{Z}/2$-graded' Pin_d^+-bundle

$$\hat{P} := P \times_{\mathrm{Spin}_d} \mathrm{Pin}_d^+ = P \sqcup P \times_{\mathrm{Spin}_d} (\mathrm{Pin}_d^+)_{odd}.$$

Define $\overline{P} = P \times_{\mathrm{Spin}_d} (\mathrm{Pin}_d^+)_{odd}$.

This defines a $\mathbb{Z}/2$-action on $\mathrm{Bord}_d^{\mathrm{Spin}}$.

Hermitian pairings on the bordism category

Let Y^{n-1} be a Spin_n-manifold, i.e. $\pi : P \to Y$ a Spin_n-bundle and $\beta : P \times_{\mathrm{Spin}_n} \mathbb{R}^n \to TY \oplus \underline{\mathbb{R}}$ a vector bundle iso.

For all $p \in P$ the vector space iso $\beta(p) : \mathbb{R}^n \to T_{\pi(p)}Y \oplus \mathbb{R}$ induces a group iso $\hat{\beta}_p : \mathrm{Pin}^+(T_{\pi(p)}Y \oplus \mathbb{R}) \to \mathrm{Pin}_n^+$ using the metric on Y.

Define $P \to \overline{P}$ by $p \mapsto p \cdot (\hat{\beta}_p(e_n))$, where $e_n \in \mathrm{Pin}(T_{\pi(p)}Y \oplus \mathbb{R})$ corresponds to the last basis vector.

This gives for all $Y \in \mathrm{Bord}_d^{\mathrm{Spin}}$, a Hermitian pairing $h_Y : Y \sqcup \overline{Y} \rightsquigarrow \emptyset$. Let P denote the collection of those Hermitian pairings.

Unitary TQFTs

Definition

A *Hermitian TQFT* is a symmetric monoidal †-functor

$$(\mathrm{Bord}_d^{\mathrm{Spin}})_P \to \mathrm{Herm}(\mathrm{sVect}).$$

A *unitary TQFT* is a symmetric monoidal †-functor

$$(\mathrm{Bord}_d^{\mathrm{Spin}})_P \to \mathrm{sHilb}\,.$$

Reflection positive TQFTs

Definition

A *reflection TQFT* is a $\mathbb{Z}/2$-equivariant TQFT $\mathrm{Bord}_d^{\mathrm{Spin}} \to \mathrm{sVect}$.
A *reflection-positive TQFT* is a reflection TQFT Z such that for all $Y \in \mathrm{Bord}_d^{\mathrm{Spin}}$, the super Hermitian pairing on $Z(Y)$ induced by h_Y is positive.

Theorem (S.)

The groupoid of reflection TQFTs is equivalent to the groupoid of Hermitian TQFTs.
The groupoid of reflection positive TQFTs is equivalent to the groupoid of unitary TQFTs.

Finding the proof to spin-statistics

If Y^{d-1} is a space, we compute mapping tori partition functions

$$Z(Y \times S^1_{per}) = \mathrm{tr}_{s}(\mathrm{id}_{Z(Y)}) = \mathrm{tr}(-1)^F_{Z(Y)}$$

$$Z(Y \times S^1_{ap}) = \mathrm{tr}_{s}\, Z((-1)^{2s}_Y) = \mathrm{tr}\left((-1)^F_{Z(Y)} Z((-1)^{2s}_Y)\right)$$

Lemma

$Z(Y \times S^1_{ap}) = \dim_{ungr} Z(Y) \iff$ *Z has spin-statistics connection*

Proof.

$(-1)^F_{Z(Y)}$ and $Z((-1)^{2s}_Y)$ commute $\Rightarrow$ simultaneous eigenbasis.

$$\dim Z(Y) = \mathrm{tr}\left((-1)^F_{Z(Y)} Z((-1)^{2s}_Y)\right)$$

$$\iff (-1)^F_{Z(Y)} Z((-1)^{2s}_Y) = \mathrm{id}_{Z(Y)}\,.$$

□

Fermionic †-compactness

In super Hilbert spaces, the diagram

$$
\begin{array}{ccccc}
\mathbb{C} & \longrightarrow & \overline{\mathbb{C}} & \xrightarrow{h^*} & \overline{\overline{V} \otimes V} \\
\big\downarrow \mathrm{coev}_V & & & & \big\downarrow \\
V \otimes \overline{V} & \xrightarrow{(-1)^F_V \otimes \mathrm{id}_{\overline{V}}} & V \otimes \overline{V} & \longleftarrow & \overline{\overline{V}} \otimes \overline{V}
\end{array}
$$

commutes.

More generally, we say a rigid symmetric monoidal dagger category is fermionically †-compact if the above diagram commutes.

Lemma

$\mathrm{Bord}_d^{\mathrm{Spin}}$ *is fermionically †-compact.*

Proof of the spin-statistics theorem

Being fermionically †-compact is equivalent to the two notions of trace differing by the $B\mathbb{Z}/2$-action:

$$\begin{array}{ccc} 1 & \xrightarrow{\mathrm{ev}^\dagger} & \overline{x} \otimes x \\ \downarrow \mathrm{coev} & & \downarrow \sigma_{\overline{x},x} \\ x \otimes \overline{x} & \xrightarrow{(-1)^F_x \otimes \mathrm{id}_{\overline{x}}} & x \otimes \overline{x} \end{array} \tag{1}$$

For a unitary fermionic TQFT Z, we get

$$\begin{aligned} Z(Y \times S^1_{ap}) &= Z(\mathrm{ev}_Y) \circ Z((-1)^{2s}_Y \sqcup \mathrm{id}_{\overline{Y}}) \circ Z(\sigma) \circ Z(\mathrm{coev}_Y) \\ &= Z(\mathrm{ev}_Y) Z(\mathrm{ev}^\dagger_Y) = \mathrm{ev}_{Z(Y)}\, \mathrm{ev}^\dagger_{Z(Y)} \\ &= \dim_\dagger Z(Y) = \dim_{ungr} Z(Y) \end{aligned}$$

Outlook

O-equivariance

- We defined unitarity using $\mathbb{Z}/2$-equivariance and '$\mathbb{Z}/2$-positivity'
- We defined spin-statistics using $B\mathbb{Z}/2$-equivariance
- $\pi_{\leq 1} O = \mathbb{Z}/2 \times B\mathbb{Z}/2$ and so for at most once extended TQFTs, O-equivariance is equivalent to $\mathbb{Z}/2 \times B\mathbb{Z}/2$-equivariance

With Cameron Krulewski and Lukas Müller, I proved a higher spin-statistics theorem for invertible TQFTs:

Theorem

For invertible TQFTs with target $I\mathbb{Z}$ the Anderson dual of the sphere, unitarity implies O-equivariance

Here unitarity is defined as $\mathbb{Z}/2$-equivariant and $\mathbb{Z}/2$-positive.

O-†-categories

We define a stronger version of unitarity by requiring O-equivariance and 'O-positivity' using 'O-† categories'.

Question

1. Is there a physical reason to do so?
2. Is O-positivity automatic given $\mathbb{Z}/2$-positivity for certain targets?

Conjecture

Every O-† 1-category is fermionically † compact.