A C^*-algebra — involution, *-homomorphisms

Def A <u>$\mathbb{Z}_2$-grading</u> is $\gamma \in \operatorname{Aut} A$ s.t. $\gamma^2 = \mathrm{id}_A$

$\Rightarrow A = A_0 \oplus A_1$ — Sum of Banach spaces

$\gamma = 1 \quad \gamma = -1$ — $A_1 \cdot A_1 \subseteq A_0$ even part is a subalgebra

$|a_0| = 0, \ |a_1| = 1$ — degree

<u>Examples</u> 1. trivial grading $\gamma = \mathrm{id}_A$ — purely even

2. pick $u \in M(A)$ $u^* = u = u^{-1}$ — self-adjoint unitary

$\gamma = \mathrm{Ad}_u$ is called an <u>even</u> grading — terrible name

3. $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ $\mathbb{Z}_2$-graded Hilbert space — orthogonal decomposition

even

$$\Rightarrow B(\mathcal{H}) = \left\{ \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} \right\}$$

odd

It is even for $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

4. $\mathbb{C}l_{-p} := \underline{\mathbb{C}[e_1, \dots, e_p]}$ — $\mathbb{C}l_{-1} \cong \mathbb{C} \oplus \mathbb{C}$

$(e_1^2 = \dots = e_p^2 = -1, \ e_i e_j + e_j e_i = 0 \ \ i \neq j)$ — Squares don't matter over $\mathbb{C}$

with e_i odd, $e_i^* = -e_i$

intertwines grading

Lemma $Cl_{-2} \cong B(\mathbb{C}^{1|1})$ where $\mathbb{C}^{p|q} = \mathbb{C}^p \oplus \mathbb{C}^q$

Proof $e_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. $\square$

Def The algebraic graded tensor product of A, B $\mathbb{Z}_2$-graded is

$$A \hat{\otimes}_{alg} B = (A_0 \otimes_{alg} B_0 \oplus A_1 \otimes_{alg} B_1) \oplus (A_1 \otimes_{alg} B_0 \oplus A_0 \otimes_{alg} B_1)$$

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \hat{\otimes} b_1 b_2$$

$$(a \hat{\otimes} b)^* = (-1)^{|a||b|} a^* \hat{\otimes} b^*$$

Spatial $\otimes$ non koszul

GNS

Recall: $\phi : A \to \mathbb{C}$ state $\Rightarrow \langle a, b \rangle_\phi := \phi(a^* b)$ Sesquilinear non negative degenerate

$I_\phi := \{ a : \langle a, a \rangle_\phi = 0 \}$ left ideal

$\Rightarrow \overline{\frac{A}{I_\phi}}$ rep ρ_ϕ of A preserving $*$

show $\exists \phi(a) \neq 0$

$\Rightarrow A \hookrightarrow B\left(\bigoplus_{\phi \text{ state}} \overline{A/I_\phi} \right)$ $*$-embedding $\Rightarrow$ define spatial $\otimes$

Problem: A graded $\Rightarrow \frac{A}{I_\phi}$ not always graded

Def ϕ is homogeneous if $\phi(A_1) = 0$

$\Rightarrow A \to B\left(\frac{A}{I_\phi}\right)$ graded

Lemma $\forall a \in A$ non zero $\exists \phi$ homogeneous s.t. $\rho_\phi(a) \neq 0$

Proof Wlog a homogeneous

a even: pick ϕ s.t. $\phi(a) \neq 0$.

Define $\phi_0(b_0 + b_1) := \phi(b_0) \Rightarrow \phi_0(a) \neq 0$

a odd: pick ϕ s.t $\phi(a^* a) \neq 0$

$\Rightarrow \phi_0(a^* a) \neq 0 \Rightarrow \rho_\phi(a^* a) \neq 0 \Rightarrow \rho_\phi(a) \neq 0$ $\square$

$$\Rightarrow \quad A \hookrightarrow B\Big(\bigoplus_{\phi \text{ homogeneous}} \overline{A/I_\phi}\Big) \quad \text{graded}$$

$$\Rightarrow A \hat{\otimes}_{alg} B \hookrightarrow B\Big(\Big(\bigoplus_{\phi \text{ homogeneous}} \overline{A/I_\phi}\Big) \hat{\otimes} \Big(\bigoplus_{\psi \text{ homogeneous}} \overline{B/I_\psi}\Big)\Big)$$

Def $A \hat{\otimes} B := \overline{A \hat{\otimes}_{alg} B}$

Concrete terrible formula for norm

$$n_p = 2^{\frac{p-2}{2}}$$

Example 1. $\mathbb{C}l_{-p} \hat{\otimes} \mathbb{C}l_{-q} \simeq \mathbb{C}l_{-(p+q)}$ $\quad n_p = 2^{\frac{p-3}{2}}$

$$\Rightarrow \mathbb{C}l_{-p} \simeq \begin{cases} B(\mathbb{C}^{n_p|n_p}) & p \text{ even} \\ B(\mathbb{C}^{n_p|n_p}) \hat{\otimes} \mathbb{C}l_1 & p \text{ odd} \end{cases}$$

2. A purely even $\Rightarrow A \hat{\otimes} \mathbb{C}l_{-1} \cong A \oplus A$

$$a \otimes e \mapsto (ia, -ia)$$

"standard odd grading"

3. A purely even $\Rightarrow A \hat{\otimes} \mathbb{C}l_{-2} \cong A \hat{\otimes} B(\mathbb{C}^{1|1})$

$$\cong M_2(A)$$

"standard even grading"

Prop A (unital) graded by Ad_u evenly, then

$$A \hat{\otimes} B \cong A \otimes B$$

$$a \hat{\otimes} b \mapsto a u^{|b|} \otimes b$$

both evenly graded $\Rightarrow A \otimes B$ evenly graded

Proof $(a_1 u^{|b_1|} \otimes b_1)(a_2 u^{|b_2|} \otimes b_2)$

$$= a_1 u^{|b_1|} a_2 u^{|b_2|} \otimes b_1 b_2$$

$$= (-1)^{|a_2||b_1|} a_1 a_2 u^{|b_1|+|b_2|} \otimes b_1 b_2 = \text{image of } (-1)^{|a_2||b_1|} a_1 a_2 \hat{\otimes} b_1 b_2$$

$\Rightarrow A \hat{\otimes}_{alg} B \cong A \otimes_{alg} B$. $\phi : A \to \mathbb{C}$, $\psi : B \to \mathbb{C}$ homogeneous

$\phi \hat{\otimes} \psi \mapsto \phi \otimes \psi$ $\square$

Hilbert Modules

Def A <u>graded Hilb B-module</u> is a Hilb B-module E with $\mathbb{C}$-linear involution $\gamma_E : E \to E$ s.t.

only B_0-linear

$$\gamma_E(\eta \cdot b) = \gamma_E(\eta)\gamma_B(b)$$

$$\gamma_B \langle \eta_1, \eta_2 \rangle = \langle \gamma_E(\eta_1), \gamma_E(\eta_2) \rangle$$

i.e. $E = E_0 \oplus E_1$ not orthogonal!

s.t. $E_i B_j \subseteq E_{i+j}$ $\quad \langle E_i, E_j \rangle \subseteq B_{i+j}$

<u>Examples</u> 1. $E = B$ as a B-module with $\gamma_E = \gamma_B$

2. If (E, γ_E) graded Hilb B-mod, then

better call this ΠE because there is no op.

$E^{op} := E$ with $\gamma_{E^{op}} = -\gamma_E$ did not work for $\gamma_{B^{op}} = -\gamma_B$

$E^{op} \cong E$ is possible

3. $E = H_B$ with $\gamma_E(b_1, b_2, \dots) = (\gamma_B(b_1), \gamma_B(b_2), \dots)$

Purely even for B purely even

Def The <u>standard graded Hilbert module</u> is

$$\hat{H}_B := H_B \oplus H_B^{op}$$

Th Kasparov stabilization : E graded Hilb B-module countably generated as an ungraded B-module

Then $\hat{H}_B \oplus E \cong \hat{H}_B$ intertwines the grading

Can also talk about odd module maps

$\mathcal{L}(E)$ graded by $T \mapsto \gamma_E T \gamma_E^{-1}$ NOT even in general

$\mathcal{K}(E)$ ——— preserved

Ex 1. $B = \mathbb{C}$, $E = \mathcal{H}_0 \oplus \mathcal{H}_1$ ($\mathbb{Z}_2$ gr Hilb space), $\mathcal{L}(E) =$ as before

2. $E = B$ (graded) $\Rightarrow \mathcal{L}(E) = M(B)$

with $\gamma_{M(B)}$ uniquely extending γ_B

Def $\phi : A \to \mathcal{L}(E_2)$ graded $*$-hom E_1 A-module

E_2 B-module. graded internal $\otimes$

$$E_1 \hat{\otimes}_\phi E_2 := E_1 \otimes_\phi E_2$$

with obvious grading

$\underline{Ex}$ $E = \hat{H}_B \cong (\mathcal{H} \oplus \mathcal{H}^{op}) \hat{\otimes}_\phi B$ ← standard graded Hilbert $\mathbb{C}$-module

← what is this if B not unital?
$\phi: \mathbb{C} \to M(B) \subseteq \mathcal{L}(E_2)$

$$\mathcal{L}(E) \cong M(B \hat{\otimes} \mathcal{K}(\mathcal{H} \oplus \mathcal{H}^{op}))$$ stable multiplier

$$\mathcal{K}(E) \cong B \hat{\otimes} \mathcal{K}(\mathcal{H} \oplus \mathcal{H}^{op})$$

$\underline{Def}$ The <u>external graded</u> $\otimes$ is

$E_1 \hat{\otimes}_{alg} E_2$ is an $A \hat{\otimes}_{alg} B$ - module : with terrible signs

$$(\eta_1 \hat{\otimes} \eta_2) \cdot (a \hat{\otimes} b) = (-1)^{|\eta_2||a|} \eta_1 \cdot a \hat{\otimes} \eta_2 \cdot b$$

with $A \hat{\otimes}_{alg} B$ - valued NON-koszul sign

$$\langle \eta_1 \hat{\otimes} \eta_2 , \xi_1 \hat{\otimes} \xi_2 \rangle = (-1)^{|\eta_2|(|\eta_1| + |\xi_1|)} \langle \eta_1, \xi_1 \rangle_{E_1} \hat{\otimes} \langle \eta_2, \xi_2 \rangle_{E_2}$$

As before $E_1 \hat{\otimes} E_2 := \overline{E_1 \hat{\otimes}_{alg} E_2}$ is

a graded $A \hat{\otimes} B$ - module

intuition for signs? It works for $E_1 = A$, $E_2 = B$:

$$(a_1 \otimes b_1)(a_2 \otimes b_2)^* = (-1)^{|a_2||b_2|} (a_1 \otimes b_1)(a_2^* \otimes b_2^*)$$

$$= (-1)^{|a_2||b_2| + |b_1||a_2|} a_1 a_2^* \otimes b_1 b_2^*$$

A ungraded, $F_0 : E_0 \to E_1$ Fredholm

$$\Rightarrow F := \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathcal{L}(E_0 \oplus E_1^{op})$$ graded Hilb module

odd, self-adjoint & Fredholm   converse

$\ker F = \ker F_0 \oplus \ker F_0^*$ graded submodule   Compact perturbation

$$\operatorname{index} F_0 = [(\ker F)_0] - [(\ker F)_1] \in K_0(A)$$