Lie Groupoids: Representations & Equivalence

Lie Groupoids Review

Composable morphisms

$$G_2 := G_1 \, {}_s\!\times_t G_1 = \{ (g,h) : s(g) = t(h) \}$$

$$\downarrow N$$

Pull back in Man

$$\mathcal{G} := \begin{tikzcd} G_1 \arrow[loop left, "i"] \arrow[d, shift left=2, "t"] \arrow[d, shift right=2, "s"'] \\ G_0 \arrow[u, "u"'] \end{tikzcd}$$

submersions

$$\mathcal{G} = \begin{tikzcd} G_1 \arrow[d, shift left=2] \arrow[d, shift right=2] \\ G_0 \arrow[u] \end{tikzcd}$$

Remark It is often <u>not</u> assumed that G_1 is Hausdorff

Def The <u>isotropy group</u> at $x \in G_0$ is

$$G_x := \{ g \in G_1 : s(g) = t(g) = x \} \subseteq G_1$$

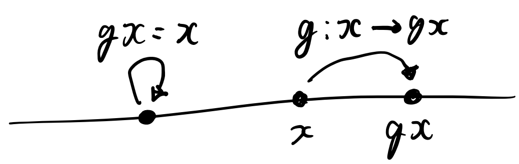
The <u>orbit</u> at $x \in G_0$ is

immersed submanifold

$$G_1 \cdot x := \{ y \in G_0 : \exists g : x \to y \} \subseteq G_0$$

Example $G \curvearrowright M$ smooth action $\Rightarrow$ $G \ltimes M$ action groupoid

$$\begin{tikzcd} G \times G \times M \arrow[d] \\ G \times M \arrow[loop left] \arrow[d, shift left=2] \arrow[d, shift right=2] \\ M \arrow[u] \end{tikzcd}$$



manifolds with equivalent points in which we remember how they are equivalent

Example $q : M \to N$ surjective submersion

$N = pt \Rightarrow$ pair grpd

$$\Rightarrow R(q) := M \times_q M = \{(x,y) : q(x) = q(y)\}$$

$N = M \Rightarrow M$ as a groupoid

$$\begin{tikzcd} R(q) \arrow[d, shift right=3, "pr_2"'] \arrow[d, shift left=3, "pr_1"] \\ M \arrow[u, "\Delta"] \end{tikzcd}$$

$$(x,y)\cdot(y,z) = (x,z)$$

Conversely, a closed embedded equivalence relation

$R \subseteq M \times M$

is of this form.

orbits are equivalence classes

<u>Example</u> (Čech groupoid) $\{U_i\}_{i\in I}$ open cover of M

coproduct = formal disjoint union

$$\begin{tikzcd} \coprod_{k,l\in I} U_k \cap U_l \ni (x,l,k) \text{ with } x \in U_k \cap U_l \arrow[d, shift right=3] \arrow[d, shift left=3] \\ \coprod_{i\in I} U_i \ni (x,i) \text{ with } x \in U_i \arrow[u] \end{tikzcd}$$

related to Čech cohomology

$$(x,l,k) : (x,k) \to (x,l)$$

$$(x,m,l)(x,l,k) = (x,m,k)$$

sorry, bad notation

$$u(x,i) = (x,i,i) \qquad (x,l,k)^{-1} = (x,k,l)$$

<u>Example</u> $E \to M$ vector bundle

The <u>general linear groupoid</u> is

$$GL(E) = \{T : E_x \xrightarrow{\sim} E_y \text{ linear isomorphisms} : x, y \in M\}$$

$$\begin{tikzcd} GL(E) \arrow[d, shift right, "t(T)=y"'] \arrow[d, shift left, "s(T)=x"] \\ M \end{tikzcd}$$

Isotropy at x is $GL(E_x)$

Actions & Representations

<u>Def</u> $q : P \to G_0$ surjective

A <u>left action</u> of $\mathcal{G} = (G_1 \times_{G_0} G_1 \to G_1 \rightleftarrows G_0)$

on P with anchor / momentum q is a map

$$\begin{tikzcd}
G_1 \times_{G_0} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[d, bend right=60, "\alpha"'] & G_1 \arrow[d, bend right, "s"'] \\
P \arrow[r, "q"] & G_0
\end{tikzcd}$$

$\alpha(g, p) =: g \cdot p$

s.t.

1. $q(g \cdot p) = t(g)$
2. $u(q(p)) \cdot p = p$
3. $g(g' \cdot p) = (g \cdot g') \cdot p$ if it makes sense

"$q(p)$ is the target of p & it has no source"

<u>Note</u>: $g : x \to y \Rightarrow g \cdot \_ : q^{-1}(x) \to q^{-1}(y)$

<u>Example</u> $G_n := G_1 \times_{G_0} \cdots \times_{G_0} G_1$ n composable morphisms

nerve

$$g \cdot (g_1, \ldots, g_n) = (g g_1, \ldots, g_n)$$

action with momentum map $q(g_1, \ldots, g_n) = t(g_1)$

Def A <u>representation</u> of $\mathcal{G} = (G_1 \times_{G_0} G_1 \to G_1 \rightrightarrows G_0)$ is a linear left action with momentum map

$q : E \to G_0$

a vector bundle, so for $g : x \to y$,

$g \cdot - : E_x \to E_y$

is linear.

Example $M \times M \rightrightarrows M$ pair groupoid

equivalence relation in which everything is equivalent

A representation is a trivial vector bundle

$M \times \mathbb{C}^n \to M$

Example A representation of $G \ltimes M$ is an equivariant vector bundle

$(E \to M, \alpha : G \times E \to E)$

i.e. $\alpha(g)_x : E_x \to E_{gx}$ linear

s.t. $\alpha(g)_{hx} \circ \alpha(h)_x = \alpha(gh)_{ghx}$

Subexample $M = \{*\} \Rightarrow$ f.d. representation of G

Subexample $G = 1 \Rightarrow$ vector bundle over M

<u>Subexample</u> $E = TM$ is canonically a representation of $G \ltimes M$ by

$$\alpha(g) = d(g \cdot \_ : M \to M)$$

<u>Recall</u> A homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ consists of smooth

$\varphi_0 : G_0 \to H_0$, $\varphi_1 : G_1 \to H_1$ compatible

with the structure maps

<u>Example</u> A representation is the same as a Lie groupoid homomorphism

$$\begin{tikzcd} G_1 \arrow[r, "\alpha"] \arrow[d, shift right] \arrow[d, shift left] & GL(E) \arrow[d, shift right] \arrow[d, shift left] \\ G_0 \arrow[u] \arrow[r, "id"] & G_0 \arrow[u] \end{tikzcd}$$

<u>Example</u> A homomorphism between action groupoids

$$\varphi : M \rtimes G \to N \rtimes H$$

consists of

$\varphi_0 : M \to N$ smooth, $f : G \to H$ group homomorphism

such that

$$\varphi_0(g \cdot m) = f(g) \cdot \varphi_0(m)$$

Notions of Equivalence

Representations up to homotopy

stacky quotient

The notion of homomorphism of Lie groupoids is 'too strict' from some viewpoints.

$\Rightarrow$ replace isomorphisms by weaker forms of equivalence

<u>Recall</u> For categories, the notion of isomorphism is too strict. Instead, if $F : \mathcal{C} \to \mathcal{D}$ is a functor, it is an equivalence if

natural isom.

$$\exists G : \mathcal{D} \to \mathcal{C} \text{ s.t. } G \circ F \simeq id,\ F \circ G \simeq id$$

$$\Updownarrow$$

F is fully faithful & essentially surjective

i.e. {categories} is a 2-category

<u>Def</u> A <u>natural transformation</u> $\varphi \Rightarrow \psi$ between Lie groupoid homomorphisms is a smooth map $T : G_0 \to H_1$ s.t.

$$\begin{tikzcd} G_1 \arrow[r, bend left, "\psi_1"] \arrow[r, bend right, "\varphi_1"'] & H_1 \end{tikzcd}$$

$T(x) : \varphi_0(x) \to \psi_0(x)$

& $\forall g : x \to y \in G_1$ the square

$$\begin{tikzcd} G_0 \arrow[r, bend left, "\psi_0"] \arrow[r, bend right, "\varphi_0"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} \varphi_0(x) \arrow[r, "T(x)"] \arrow[d, "\varphi_1(g)"'] & \psi_0(x) \arrow[d, "\psi_1(g)"] \\ \varphi_0(y) \arrow[r, "T(y)"] & \psi_0(y) \end{tikzcd}$$

Commutes.

<u>Example</u> $G_1 = G$, $G_0 = * = H_0$, $H_1 = H$ Lie group as a Lie groupoid.

A natural transformation between Lie group maps

$$\begin{tikzcd} G \arrow[r, bend left, "\varphi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & H \arrow[from=U, to=D, Rightarrow] \end{tikzcd}$$

is an $h \in H$ s.t. $\varphi(g) = h\,\psi(g)\,h^{-1}$ $\forall g \in G$

<u>Def</u> A morphism $\varphi : \mathcal{G} \to \mathcal{H}$ of Lie groupoids is a <u>strong equivalence</u> if $\exists\ \psi : \mathcal{H} \to \mathcal{G}$ and natural transformations

$$\varphi \circ \psi \Rightarrow id_{\mathcal{H}}, \quad \psi \circ \varphi \Rightarrow id_{\mathcal{G}}$$

strong equivalence weaker than iso

<u>Example</u> the pair groupoid is strongly equivalent to $\{pt\}$:

Fix $x_0 \in M$

$$\begin{tikzcd} M \times M \arrow[r] \arrow[d, shift left=2] \arrow[d, shift right=2] & pt \arrow[r, bend right, "{(x_0, x_0)}"] & M \times M \arrow[d, shift left=2] \arrow[d, shift right=2] \\ M \arrow[u] \arrow[r] \arrow[urr, "T"] & pt \arrow[r, bend left, "x_0"] & M \arrow[u] \end{tikzcd}$$

$T : \mathrm{id}_{M\times M} \Rightarrow pr_{x_0}$ $\quad T(x) : x \to x_0$

$$\begin{tikzcd} x \arrow[r, "T(x)"] \arrow[d] & x_0 \arrow[d] \\ y \arrow[r, "T(y)"'] & x_0 \end{tikzcd}$$

$\boxed{\text{Properness}}$

<u>Def</u> A Lie groupoid $\begin{matrix} G_1 \\ \downarrow\uparrow\downarrow \\ G_0 \end{matrix}$ is called <u>proper</u> if

$$(s,t) : G_1 \to G_0 \times G_0$$

is a proper map

(inverse image of compact is compact)

<u>Note</u> Action groupoid $\Rightarrow$ usual notion of proper action

i.p. for G discrete, proper $\Leftrightarrow$ "properly discontinuous"

Lie group proper $\Leftrightarrow$ compact

Example $E \to M$ vector bundle

$GL(E)$ is not proper.

Picking a metric on E, we can define $O(E)$, which is proper.

Th If $G \circlearrowright M$ is free & proper, then $\exists!$ smooth structure on M/G such that

$$M \to M/G \quad x \mapsto [x]$$

is a submersion. This makes $[\cdot]$ into a principal G-bundle. Conversely, if

$$M \to B$$

is a principal bundle, $G \circlearrowright M$ is free & proper and $B \cong M/G$ compatibly.

Example $G \circlearrowright M$ free proper

Then $G \ltimes M$ is strongly equivalent to M/G iff $M \to M/G$ is the trivial principal bundle

Suppose strongly equiv

$$\begin{tikzcd}
G \times M \arrow[r] & M/G \arrow[r] & G \times M \arrow[d, shift right=2] \arrow[d, shift left=2] \\
M \arrow[u] \arrow[r, "\pi"] \arrow[urr, "T"] & M/G \arrow[r, "s"] & M \arrow[u]
\end{tikzcd}$$

$$\underset{\substack{\uparrow \\ G}}{T(x)} : x \to s \circ \pi(x) \Rightarrow T(x)\,x = s \circ \pi(x)$$

$$\Rightarrow [s \circ \pi(x)] = [x]$$

$$\Rightarrow \sigma[x] := s \circ \pi(x)$$

Won't show the converse

1. is a section of $M \xrightarrow{[\cdot]} M/G$
2. is well-defined because π is a morphism of Lie groupoids

$\Rightarrow$ $M \to M/G$ is the trivial principal bundle

<u>Idea</u> $M \rtimes G$ is a well-behaved generalization of M/G for the case when $G \curvearrowright M$ is not free & proper

$\Rightarrow$ want an even weaker notion of equivalence so that $M \rtimes G \sim M/G$ for free & proper actions

fully faithful & ess surj works

$\boxed{\text{Weak Equivalence}}$

Def A morphism $\varphi: \mathcal{G} \to \mathcal{H}$ of Lie groupoids is <u>essentially surjective</u> if

$$\begin{tikzcd}
G_0 {}_{\varphi_0}\times_s H_1 \arrow[r] \arrow[d, "\sim"'] \arrow[rd, bend left=60, "t \circ pr_2"] & H_1 \arrow[d, "s"'] \arrow[d, bend left, "t"] \\
G_0 \arrow[r, "\varphi_0"] & H_0
\end{tikzcd}$$

$t \circ pr_2$ is a surjective submersion

$$\left(\forall y \in H_0 \ \exists x \in G_0 \ \exists h: \varphi_0(x) \to y \quad t(h) = y, \ s(h) = \varphi_0(x) \right)$$

A morphism $\varphi: \mathcal{G} \to \mathcal{H}$ is <u>fully faithful</u> if

$$\begin{tikzcd}
Mor_{\mathcal{G}}(x,y) \arrow[r, "\varphi_1"] \arrow[d] & Mor_{\mathcal{H}}(\varphi_0(x), \varphi_0(y)) \arrow[d] \\
G_1 \arrow[r, "\varphi_1"] \arrow[d, "(s,t)"'] & H_1 \arrow[d, "(s,t)"] \\
G_0 \times G_0 \arrow[r, "(\varphi_0,\varphi_0)"] & H_0 \times H_0
\end{tikzcd}$$

is a pullback square

A morphism is a <u>weak equivalence</u> if it is fully faithful & essentially surjective.

Example $\{U_i\} \subseteq M$ open cover.

The Čech groupoid is weakly equivalent to M

$$\begin{tikzcd} \coprod_{k,\ell \in I} U_k \cap U_\ell \arrow[r] \arrow[d, shift right=2] \arrow[d, shift left=2] & M \\ \coprod_{i \in I} U_i \arrow[u] \arrow[r] & M \end{tikzcd}$$

$$\mathrm{Mor}((x,i),(y,j)) = \begin{cases} \emptyset & x \neq y \\ \{(x,j,i)\} & x = y \end{cases}$$

Note A Lie groupoid homomorphism $M \to pt \rtimes G$ is necessarily trivial.

A Lie groupoid homomorphism $\check{C}ech(U_i) \to pt \rtimes G$ is a collection $\varphi_{k,\ell} : U_k \cap U_\ell \to G$ such that $\varphi_{k,\ell} \cdot \varphi_{\ell,m}(x) = \varphi_{k,m}(x)$ $\forall x \in U_k \cap U_\ell \cap U_m$

Čech 1-cocycle, principal G-bundle

Def Lie groupoids $\mathcal{G}, \mathcal{H}$ are called <u>Morita equivalent</u> if there is a third Lie groupoid $\mathcal{H}'$ & weak equivalences

$$\begin{tikzcd} & \mathcal{H}' & \\ \mathcal{G} \arrow[ur, "\varphi", "\sim"'] & & \mathcal{H} \arrow[ul, "\psi"', "\sim"] \end{tikzcd}$$

Proposition This is an equivalence relation

Def A <u>generalized morphism</u> $\mathcal{G} \to \mathcal{H}$ consists of a Lie groupoid $\mathcal{H}'$ and homomorphisms

$$\begin{tikzcd} & \mathcal{H}' & \\ \mathcal{G} \arrow[ur, "\varphi", "\sim"'] & & \mathcal{H} \arrow[ul, "\psi"'] \end{tikzcd}$$

with φ an equivalence

Prop Every generalized morphism $\mathcal{G} \to \mathcal{H}$ can be realized by a "Čech refinement"

$$\mathcal{G} \leftarrow \mathcal{G}_{\{U_i\}} \longrightarrow \mathcal{H} \qquad\qquad U_i \subseteq G_0$$

Th Identify two generalized morphisms if they have a common refinement:

$$
\begin{tikzcd}
& \mathcal{H}'' & \\
& \mathcal{H}' \arrow[u] & \\
\mathcal{G} \arrow[ur, "\sim", "\varphi"'] \arrow[uur, bend left] & & \mathcal{H} \arrow[ul, "\psi"] \arrow[uul, bend right]
\end{tikzcd}
$$

The resulting category is equivalent to the category of stacks on the site **Man** of smooth manifolds (sheaves on **Man** with values in groupoids)

Orbifold Aside

Def A Lie groupoid is called <u>étale</u> when

$s : G_1 \to G_0$

is a local diffeomorphism.

An <u>orbifold</u> is a proper étale Lie groupoid

<u>I think</u>

- every orbifold is locally isomorphic to $\mathbb{R}^n /\!\!/ H$ with H finite
- every orbifold is globally isomorphic to $N /\!\!/ H$ with $H \circlearrowright N$ a smooth action with finite stabilizers

Th Let $\mathcal{C}_{\mathcal{G}}$ be the category of left $\mathcal{G}$-torsors.

There is a functor

$$\mathcal{C}_{\mathcal{G}} \to \mathrm{Man}$$

given by mapping $\pi: P \to B$ to B.

This is a stack over Man.

The stacks $\mathcal{C}_{\mathcal{G}}$ & $\mathcal{C}_{\mathcal{H}}$ are isomorphic iff $\mathcal{G}$ & $\mathcal{H}$ are Morita equivalent.