1. Free Fermion Symmetries

2. Charge Symmetry & Quasi-Particle Vacua

3. Particle-Hole Symmetries

4. Freed-Moore Symmetries: a comparison

5. Tenfold Ways SRFL v.s. FH: a comparison

1. Free Fermion Symmetries

<u>Recall</u>: $(-1)^F \in G \xrightarrow{\theta} \underbrace{\{\pm 1\}}_{= \mathbb{Z}_2}$ fermionic group

$\theta((-1)^F) = 1 \qquad ((-1)^F)^2 = 1$

Def A <u>free fermionic representation</u> of G consists of

1. A real Hilbert space M (fermionic mode space), which we complexify to $W := M \otimes_{\mathbb{R}} \mathbb{C}$ (Nambu space) with real structure γ

   $\gamma R(g) \gamma = R(g)$

2. A (strongly cts) homomorphism $R: G \to O(M)$ which we extend to a $R: G \to U(W) \amalg AU(W)$ (anti-unitary operators)

   s.t. $R((-1)^F) = -\mathrm{id}$ and

   $R(g) \in AU(W) \Leftrightarrow \theta(g) = -1$

   $R(g)(m_1 + i m_2) = R(g)(m_1) - i R(g)(m_2)$

---

2. Charge Symmetry & Quasi-Particle Vacua

Ex/Def A <u>charge symmetry</u> is a free fermionic representation of

$G = \mathbb{Z}_4$ with nontrivial $(-1)^F$ but trivial θ

$\Rightarrow$ orth. $\mathbb{C}$-structure on M extended $\mathbb{C}$-linearly to $I = iQ : W \to W$ (charge)

$\Rightarrow W = V \oplus \gamma(V)$ ± 1 eigenspaces of Q.

V: style electron Hilbert space; $\gamma(V)$: holes

... charge relation

Fix a Q from now on.

Ex Last example gives a free fermionic rep of $U(1)$ with nontrivial $(-1)^F = -1$

spin-charge

$\Updownarrow$

as $e^{i\varphi Q} = \begin{pmatrix} e^{i\varphi} & \\ & e^{-i\varphi} \end{pmatrix} \begin{matrix} V \\ \gamma(V) \end{matrix}$

Not 1-1, since particles of odd charge > 1 are not allowed

$\gamma J \gamma = J \Rightarrow \gamma H \gamma = -H$

Def A quasi-particle vacuum (QPV) J $\left(= i\frac{H}{|H|}\right)$ is also a real orthogonal complex structure on M that we extend $\mathbb{C}$-linearly to W

~ flattened gapped Hamiltonian

Rmk: morally very different

Def In a free fermionic rep (G, W, γ, R), J is G-invariant if

$$R(g)\, J = \theta(g)\, J\, R(g) \qquad \left([H, R(g)] = 0 \right)$$

Idea: $\pi_0(\text{IQPVs})$ = free fermion topological phases

3. Particle-Hole Symmetry

<u>Note</u> A real $T : W \to W$ (e.g. some $R(g)$)
commutes with Q iff

$$T = \begin{pmatrix} T|_V & 0 \\ 0 & \gamma T|_V \gamma \end{pmatrix}$$

and T is (anti) unitary iff $T|_V$ (anti) unitary

<u>Def</u> A <u>particle-hole symmetry</u> is a real $K : W \to W$ (e.g. some $R(g)$)
which anti commutes with Q $\quad K = R(g)$

<u>Note</u> K PHS iff of the form

$$K = \begin{pmatrix} 0 & \Gamma\gamma \\ \gamma\Gamma & 0 \end{pmatrix} \begin{matrix} V \\ \gamma(V) \end{matrix}$$

exchanges particles & holes

for some $\Gamma : V \to V$. Moreover,
Γ is anti-unitary iff K is unitary

4. Freed & Moore's Symmetries: a Comparison

<u>Def</u> A <u>Freed-Moore group</u> consists of

1. a Lie group G^τ
2. two cts homomorphisms $\phi, c : G^\tau \to \mathbb{Z}_2$
3. A subgroup $U(1) \subseteq G^\tau$

s.t. $e^{it} g = g\, e^{\phi(g) it}$ $\forall e^{it} \in U(1)$.

$c(g) = -1$
$\Downarrow$
$R(g) Q = -Q R(g)$

It becomes a fermionic group via $(-1)^F := -1 \in U(1) \subseteq G^\tau$
and $\theta := \phi \cdot c$

$U(1)$ = charge $\quad c(g) = \pm 1$ Particle hole reversing / preserving

$\phi(g) = \pm 1$ (anti) unitary on single particle space V

Def A Freed-Moore rep of a Freed-Moore group on a $\mathbb{Z}_2$-graded $\mathbb{C}$-Hilbert space $V = V_+ \oplus V_-$ is a cts

conduction electrons (V_+) $\quad$ valence electrons (V_-)

$\rho : G^\tau \to U(V) \amalg AU(V)$

- $U(1) \subseteq G^\tau$ acts as scalars
- $\rho(g)$ is unitary / anti-unitary depending on ϕ
- $\rho(g)$ is even / odd depending on c ← "$\rho(g)$ commutes v.s. anticommutes with H"

Prop A Freed-Moore rep (V, ρ) is equivalent to a free fermionic rep W with a charge symmetry Q s.t. $U(1) \subseteq G^\tau$ acts by the charge symmetry together with a G^τ-IQPV J.

Equivalence of cats

Pf (sketch)

$$R(g) = \begin{cases} \begin{pmatrix} \rho(g) & \\ & \gamma\rho(g)\gamma \end{pmatrix} \begin{matrix} V \\ \gamma(V) \end{matrix} & c(g) = +1 \\ \begin{pmatrix} & \rho(g)\gamma \\ \gamma\rho(g) & \end{pmatrix} \begin{matrix} V \\ \gamma(V) \end{matrix} & c(g) = -1 \end{cases}$$

then $R(g)$ is unitary if $\phi(g)c(g) = +1$

& $R(g)$ is anti-unitary if $\phi(g)c(g) = -1$

The $\mathbb{Z}_2$-grading $V = V_+ \oplus V_-$ gives

a $\mathbb{Z}_2$-grading $W = (V_+ \oplus \gamma(V_-)) \oplus (V_- \oplus \gamma(V_+))$ $\left(\sim \frac{H}{|H|}\right)$

and all $R(g)$ are even in this grading, e.g. $R(g)J = \theta(g)JR(g)$.

$\Leftrightarrow$ $[R(g), H] = 0 \quad \forall g$

$\square$

Talk 2: PHS, 2^{nd} Quantization & ten-fold ways

2^{nd} Quantization

W $\mathbb{C}$ Hilbert space (Nambu space)

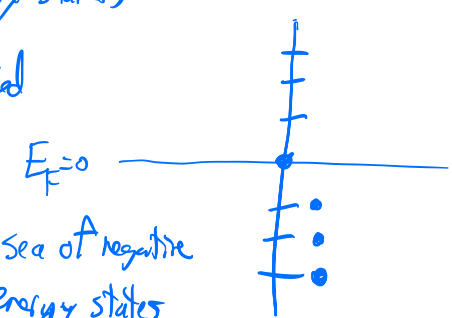
interchanges bra & ket

real structure γ; anti-unitary $\gamma^2 = 1$ (Particle-hole conjugation)

$H : W \to W$ self-adjoint imaginary ($\gamma H \gamma = -H$) s.t. $0 \notin \operatorname{Spec} H$ (gapped Hamiltonian)

$W_+ :=$ spectral projection onto $(0, \infty)$ (positive energy states)

Fock space $\mathcal{F} = \Lambda^\bullet W_+$ $\Rightarrow$ $|0\rangle$ = all valence bands filled



How to 2^{nd} Q operators on W to $\mathcal{F}$?

$T : W \to W$ real (anti)unitary operator; $\gamma T \gamma = T$ s.t. $[T, H] = 0$ (Symmetry)

$\hat{T}(w_1 \wedge \dots \wedge w_n) = T w_1 \wedge \dots \wedge T w_n$ + complete

Works because $T(W_+) \subseteq W_+$.

$T \mapsto \hat{T}$ is a homomorphism & T (anti-) unitary $\Leftrightarrow$ $\hat{T}$ (anti-) unitary

$\sum_{n=1}^{\infty} \lambda_n^2 < \infty$

<u>Note</u>: Can only 2^{nd} Q. operator $T: W \to W$ if $\left[T, \frac{H}{|H|} \right]$ is Hilbert-Schmidt

<u>Ex</u> γ cannot be 2^{nd} Q. unless $\dim W_+ < \infty$, since

$\hat{\gamma}(|0\rangle) = z \; e_1 \wedge e_2 \wedge e_3 \wedge \ldots \notin F$

where $z \in U(1)$ & $\{e_i\} \subseteq W_+$ orthonormal basis

If $\dim W_+ = N < \infty$, a 2^{nd} Q of γ is a choice of volume form

& $\hat{\gamma}: \Lambda^n W_+ \longrightarrow \Lambda^n W_+^* \xrightarrow{\text{Hodge} *} \Lambda^{N-n} W_+$

"wedge isomorphism"

$K = \gamma \Gamma_{\mathbb{R}}$ anticommutes with H

Talk 3 : Ten-fold ways compared

1. Freed - Hopkins & Kreck-Stolz-Teichner
2. Schnyder-Ryu-Furusaki-Ludwig & Freed-Moore
3. Comparison between 1 & 2 & Kennedy-Zirnbauer

1. FH & KST's 10-fold way

<u>Def</u> A <u>superdivision algebra</u> D is a superalgebra s.t. all homogeneous elements are invertible.

<u>Ex</u> $D = \mathbb{C}l_1 = \dfrac{\mathbb{C}[\overset{\text{odd}}{e}]}{(e^2-1)}$ is a superdivision algebra

<u>Prop</u> * There are 10 superdivision algebras $/\mathbb{R}$

$$\mathbb{C} \quad \mathbb{C}l_1 \quad \mathbb{R} \quad Cl_{+1} \quad Cl_{+2} \quad Cl_{+3} \quad \mathbb{H} \quad Cl_{-3} \quad Cl_{-2} \quad Cl_{-1}$$

* There are 10 fermionic groups K with $\mathrm{Ker} \in \{O_1, U_1, SU_2\}$

$$K = \frac{D^\times}{\mathbb{R}_{>0}}$$

2. SRFL & FM

<u>Recall</u>: Freed-Moore groups
$$\begin{tikzcd} & & \mathbb{Z}_2 \\ U_1 \subseteq & G^\tau \arrow[ur, "c"] \arrow[r, "\phi"] & \mathbb{Z}_2 \end{tikzcd}$$
& $zg = g z^{\phi(g)}$

$\theta = \phi \cdot c$

TRS "charge conjugation" / "particle-hole conjugation"

Let $\mathcal{C}$ be the group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, T\} \times \{1, C\}$
and define homomorphisms $\theta, \phi, c : \mathcal{C} \to \mathbb{Z}_2$ by

	T	C	CT
θ	-1	1	-1
ϕ	-1	-1	1
c	1	-1	-1

CT ~ sublattice or "chiral symmetry"

<u>Def</u> A <u>CT-group</u> is a Freed-Moore group G^τ s.t. $G^\tau/\mathcal{U}_1 \subseteq \mathcal{C}$ and $\phi, c : G^\tau \to \mathbb{Z}_2$ are compatible with the above

<u>Prop</u> There are ten isomorphism classes of CT-groups given by a choice of subgroup of $\mathcal{C}$ & a choice of ± 1 for the squares of lifts of T & C

$$1 \to U(1) \to G^\tau \to G \to 1$$

Different between AZ & FH

Different between SRFL & FH

1.	K	$\mathcal{U}_1$	$\mathcal{U}_1 \times \mathbb{Z}_2^\theta$	$\mathbb{Z}_2^F$	$\mathbb{Z}_2^\theta \times \mathbb{Z}_2^F$	$\mathcal{U}_1 \rtimes \mathbb{Z}_2^\theta$	$\dfrac{SU_2 \times \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$	SU_2	$SU_2 \times \mathbb{Z}_2^\theta$	$\dfrac{\mathcal{U}_1 \rtimes \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$	$\mathbb{Z}_4^\theta$
		Q	Q, K	$-$	K	Q, $T^2 = 1$?	σ^N, T	σ^N	σ^N, K	T, Q	T

2	G^T	U_1	$U_1 \times \mathbb{Z}_2^\theta$	$U_1 \rtimes \mathbb{Z}_2$	$(\underbrace{U_1 \rtimes \mathbb{Z}_2}_{C}) \times \underbrace{\mathbb{Z}_2^\theta}_{CT}$	$U_1 \rtimes \mathbb{Z}_2^\theta$	$\frac{\frac{U_1 \rtimes \mathbb{Z}_4}{\mathbb{Z}_2} \times \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$	$\frac{U_1 \rtimes \mathbb{Z}_4}{\mathbb{Z}_2}$	$\frac{U_1 \rtimes \mathbb{Z}_4}{\mathbb{Z}_2} \times \mathbb{Z}_2^\theta$	$\frac{U_1 \rtimes \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$	$\frac{(U_1 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$
	(C,T)	00	diag	+0	++	0+	−+	−0	−−	0−	+−
3		U_1	$U_1 \times \mathbb{Z}_2^\theta$	$\mathbb{Z}_2^F$	$\frac{\frac{U_1 \rtimes \mathbb{Z}_4^\theta}{\mathbb{Z}_2} \times SU_2}{\mathbb{Z}_2} \times \mathbb{Z}_2^\theta$	$\frac{\frac{U_1 \rtimes \mathbb{Z}_4^\theta}{\mathbb{Z}_2} \times SU_2}{\mathbb{Z}_2}$	$\frac{SU_2 \times \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$	SU_2	$\frac{U_1 \rtimes \mathbb{Z}_4}{\mathbb{Z}_2} \times \mathbb{Z}_2^\theta$	$\frac{U_1 \rtimes \mathbb{Z}_4^\theta}{\mathbb{Z}_2}$	$\mathbb{Z}_4^\theta$
		Q	Q, K	–	σ^N, T, Q, K	σ^N, T, Q	σ^N, T	σ^N	T, Q, K	T, Q	T
	class	A	AIII	D	BDI	AI	CI	C	CII	AII	DIII

1. Free fermion SPT classifications probably agree because of Morita equivalences

2. Interacting fermion SPT phases probably disagree

Example class D: $$\frac{{}^{\text{“}}C^*_{\mathbb{C}}[U_1 \rtimes \mathbb{Z}_2]}{\text{“}U_1 \text{ acts regularly”}} \cong \frac{\mathbb{R}[i, C]}{(i^2+1, C^2-1, iC+Ci)} \simeq M_2(\mathbb{R})$$

only $\mathbb{R}$ C^*-alg!

Morita equiv to $$\frac{C^*_{\mathbb{R}}(\mathbb{Z}_2)}{\text{“}\mathbb{Z}_2 \text{ acts regularly”}} \cong \mathbb{R}$$

“charge & PHS cancel”

But: $SPT^3_{U_1 \rtimes \mathbb{Z}_2} \not\cong SPT^3_{\mathbb{Z}_2^F}$

because I claim $\Omega_3^{(Pin^+ \otimes (U_1 \rtimes \mathbb{Z}_2))er} = \mathbb{Z}_2$ while $\Omega_3^{Spin} = 0$