Computations in the Adams Spectral Sequence

1. Steenrod operations on cohomology
2. Some properties of the Steenrod algebra
3. The classical Adams spectral sequence & Ext
4. E_2 to compute stable stems
5. Important subalgebras of the Steenrod algebra
6. Computing ko via Adams
7. The dual Steenrod algebra & the cobar complex

1. Steenrod operations on cohomology

Everything will be over prime $p=2$

The cohomology of any space $H^*(X)$ is a commutative algebra under cup product.

$\Rightarrow$ useful to distinguish spaces up to homotopy

Example: $\mathbb{RP}^2 \not\simeq S^1 \vee S^2$

But cup product is not stable

Claim $H^*(\Sigma X)$ has trivial cup product for any space X

$\Rightarrow$ useless to distinguish spaces up to stable homotopy

However, a combination of higher cup products is: Steenrod squares

Note Cup product on $H^*(X)$ is induced by homotopy commutative product on (say singular) cochains $C^*(X)$

$$C^*(X) \otimes_{\mathbb{F}_2} C^*(X) \overset{\text{Choice!}}{\simeq} C^*(X \times X) \xrightarrow{\Delta^*} C^*(X)$$

$\exists$ chain homotopy $\cup_1 : \cup \circ \tau \simeq \cup$ of degree -1

Flip

e.g. $\forall a, b \in C^*(X)$, $a \cup b - b \cup a = d(a \cup_1 b) + da \cup_1 b + a \cup_1 db$

<u>Continue</u> $\cup_i : \cup_{i-1} \circ T \simeq \cup_{i-1}$ deg $-i$

Could define $a \in H^n(X) \Rightarrow Sq^{n-i}(a) := a \cup_i a$ well-defined

<u>Example</u> $H^*(\mathbb{RP}^\infty) \simeq \mathbb{F}_2[x]$

$|x| = 1$

$Sq^0(x) = x \quad Sq^1(x) = x^2$

$Sq^n(x) = 0$ if $n > 1$

$Sq(x^n) = Sq(x)^n = (x + x^2)^n$

$= x + nx + \binom{n}{2} x^2 + \ldots$

Cheat sheet

$Sq^n(a) = a^2$ if $|a| = n$

$Sq^n(a) = 0$ if $|a| < n$

$Sq^0(a) = a$

<u>Cartan:</u> Sq is a ring homomorphism

"$Sq^0 + Sq^1 + \ldots$

Can now distinguish $\mathbb{RP}^2$ & $S^1 \vee S^2$ stably

2. Some properties of the Steenrod algebra

<u>Recall</u> the Steenrod algebra:

a $\mathbb{Z}$-graded $\mathbb{F}_2$-algebra with degree $n \in \mathbb{Z}$ part

$$\mathcal{A}_n = \pi_{-n} \operatorname{Map}(H\mathbb{F}_2, H\mathbb{F}_2)$$

degree

$$= H^n(H\mathbb{F}_2)$$

$$= \lim_m \tilde{H}^{n+m}(K(\mathbb{Z}/2, m))$$

$$= \{\text{stable cohomology operations of degree } n\}$$

$$\mathcal{A} = \frac{\mathbb{F}_2\{Sq^1, Sq^2, Sq^3, \ldots\}}{\text{Adem relations}} \qquad |Sq^n| = n$$

<u>Def</u> An <u>augmentation</u> of an $\mathbb{F}_2$-algebra A is an algebra homomorphism $\varepsilon : A \to \mathbb{F}_2$

The <u>augmentation ideal</u> is $\operatorname{Ker} \varepsilon$

We say $a \in A$ is <u>decomposable</u> if $a \in (\operatorname{Ker} \varepsilon)^2$

<u>Example</u> group algebra $A = \mathbb{F}_2[G]$ finite

with $\varepsilon(g) = 1 \ \forall g$

Def A $\mathbb{Z}$-graded $\mathbb{F}_2$-algebra A is <u>connected</u> if $A_n = 0$ for $n < 0$ and $A_0 = \mathbb{F}_2$.

Note A connected algebra is augmented by projection onto A_0.

Example Steenrod algebra $\mathcal{A}$ is connected.

Prop $Sq^n \in \mathcal{A}$ is decomposable $\iff$ $n \neq 2^i$

Def A monomial $Sq^{i_1} \cdots Sq^{i_k} \in \mathcal{A}$ is called <u>admissible</u> if $i_j \geq 2 i_{j-1}$.

Prop Admissible monomials form a basis of $\mathcal{A}$ called the <u>Serre-Cartan basis</u>

3. The Classical Adams Spectral Sequence & Ext

If X is a spectrum, the classical Adams spectral sequence says

"internal" Ext grading   "cohomological" module grading

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}\left(H^*(X), \Sigma^t\mathbb{F}_2\right) \Rightarrow \pi_{t-s}(X)^\wedge_2$$

$\mathcal{A}$-module via augmentation

2-completion:

$\mathbb{Z}^\wedge_2 = \mathbb{Z}_2$ 2-adics

$(\mathbb{Z}/2^k)^\wedge_2 = \mathbb{Z}/2^k$

When $s \geq 1$, $\mathrm{Ext}_{\mathcal{A}}^{s,t}(M,N)$ can be represented by extensions

$$0 \to \Sigma^t N \to P_1 \to \cdots \to P_s \to M \to 0$$

modulo isomorphism.

There is a bigraded product (Yoneda product) extending composition $(f,g) \mapsto g \circ f$

$$\mathrm{Ext}_{\mathcal{A}}^{s_1,t_1}(M,L) \times \mathrm{Ext}_{\mathcal{A}}^{s_2,t_2}(L,N) \to \mathrm{Ext}_{\mathcal{A}}^{s_1+s_2,t_1+t_2}(M,N)$$

For $L = N = \mathbb{F}_2$, $M = H^*(X)$, get an "action" of the E_2 for $\mathbb{S}$ on the E_2 for X

<u>Prop</u> 1. Differentials d_r in the spectral sequence are a derivation

2. Get induced products on all E_r

3. The product on E_∞ is induced by the action $\pi_*(\mathbb{S}) \circlearrowright \pi_*(X)$

The Yoneda product can be represented by extensions for $s_1, s_2 \geq 1$:

$$0 \to \Sigma^{t_1}\mathbb{F}_2 \to P_1 \to \cdots \to P_{s_1} \to M \to 0$$

$$0 \to \Sigma^{t_2}\mathbb{F}_2 \to Q_1 \to \cdots \to Q_{s_2} \to \mathbb{F}_2 \to 0$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \Sigma^{t_1+t_2}\mathbb{F}_2 \arrow[r] & \Sigma^{t_1}Q_1 \arrow[r] & \cdots \arrow[r] & \Sigma^{t_1}Q_{s_2} \arrow[rr] \arrow[dr, dashed] & & P_1 \arrow[r] & \cdots \arrow[r] & P_s \arrow[r] & M \arrow[r] & 0 \\
& & & & & \Sigma^{t_1}\mathbb{F}_2 \arrow[ur, dashed] & & & & &
\end{tikzcd}$$

4. E_2 to compute stable stems

Example $X = \mathbb{S}$

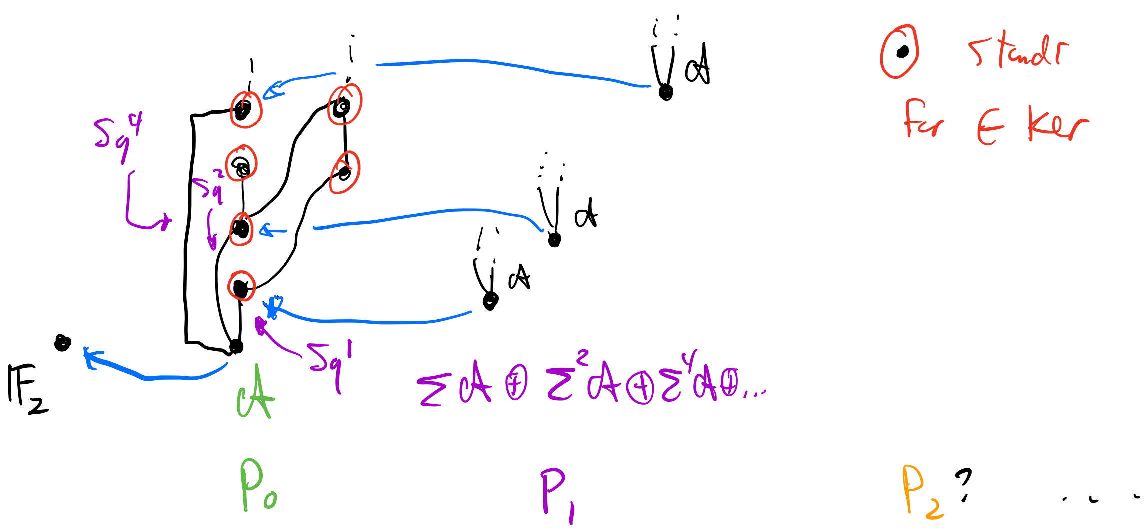
How to compute $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$?

Take a minimal free resolution

$$\mathbb{F}_2 \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

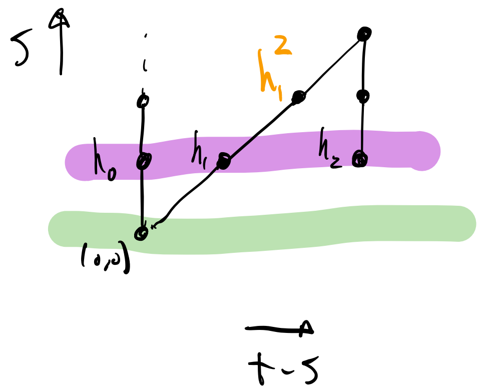
of $\mathbb{Z}$-graded $\mathcal{A}$-modules, then $\mathrm{Hom}_{\mathcal{A}}$ to $\mathbb{F}_2$:

Lemma The differentials are zero

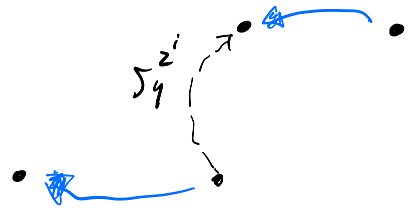


$$\mathrm{Ext}_{\mathcal{A}}^{0,t}(\mathbb{F}_2,\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & t=0 \\ 0 & t \neq 0 \end{cases} \qquad \mathrm{Ext}_{\mathcal{A}}^{1,t}(\mathbb{F}_2,\mathbb{F}_2) = \begin{cases} \mathbb{F}_2\, h_i & t = 2^i \\ 0 & t \neq 2^i \end{cases}$$

$\Rightarrow$ First two rows of E_2-page for computing stable stems 2-locally



Hopf maps h_i are extensions



$\in Ext_{\mathcal{A}}^{1,2^i}(\mathbb{F}_2,\mathbb{F}_2)$

h_i^2 related to Kervaire invariant 1 problem:

h_i^2 is a permanent cycle $\iff$ $\exists\ 2^{i+1}-2$ -dim mfd of Kervaire invariant 1

Open problem <u>only</u> for h_6^2 (dimension 126)

<u>Conclusion</u>: $Ext_{\mathcal{A}}^{s,t}$ is hard.

<u>Easier</u> $Ext_{\mathcal{B}}^{s,t}$ for $\mathcal{B} \subseteq \mathcal{A}$ finite-dimensional.

<u>But</u> is that useful?

5 Important subalgebras of the Steenrod algebra

<u>Def</u> $\mathcal{B} \subseteq \mathcal{A}$ subalgebra, $\mathcal{A}/\!/\mathcal{B} := \mathcal{A} \otimes_{\mathcal{B}} \mathbb{F}_2$

<u>Th</u> (change of rings) If N is a graded $\mathcal{A}$-module, then

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A}/\!/\mathcal{B}, N) \cong \mathrm{Ext}^{s,t}_{\mathcal{B}}(\mathbb{F}_2, N)$$

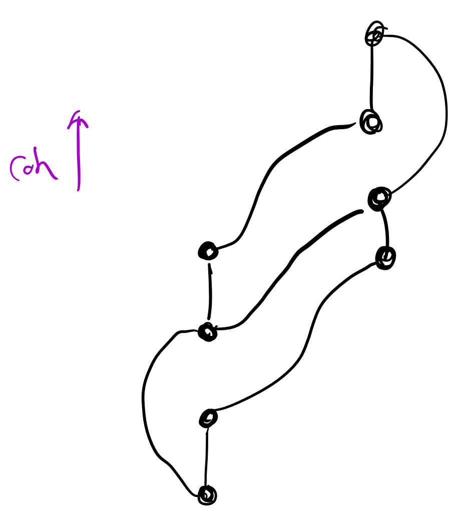
<u>Def</u> $\mathcal{A}(n) := \langle Sq^1, \ldots, Sq^{2^n} \rangle \subseteq \mathcal{A}$

$\mathcal{E}(n) := \langle Q_0, \ldots, Q_n \rangle \subseteq \mathcal{A}(n)$ where

$Q_0 := Sq^1$

$Q_i := Sq^{2^i} Q_{i-1} + Q_{i-1} Sq^{2^i}$

<u>Note</u> $\mathcal{E}(n)$ is an exterior algebra on $Q_0, \ldots, Q_n$



$\mathcal{A}(1)$

$H^*(ku) \cong \mathcal{A}/\!/\mathcal{E}_1$

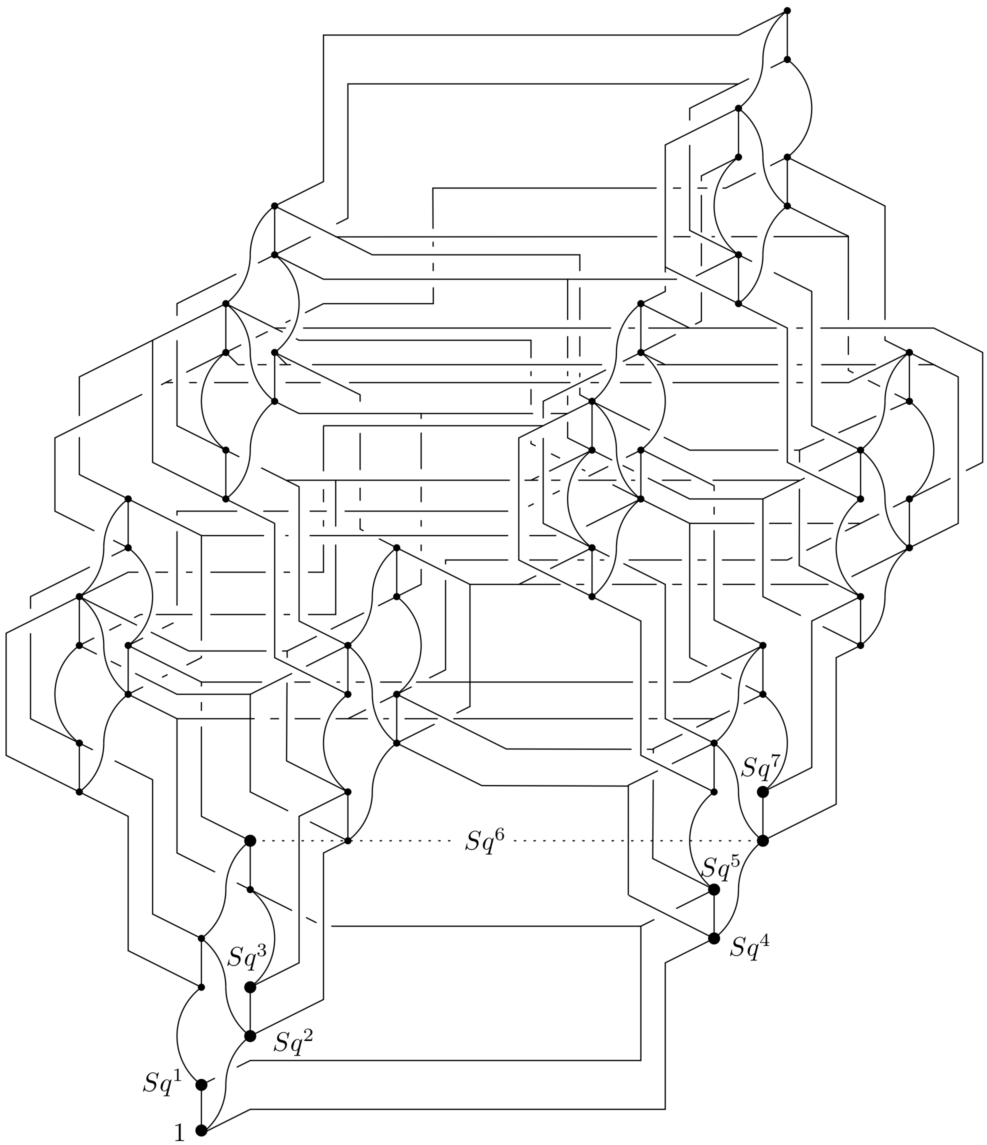
$H^*(ko) \cong \mathcal{A}/\!/\mathcal{A}_1$

$H^*(tmf) \cong \mathcal{A}/\!/\mathcal{A}_2$

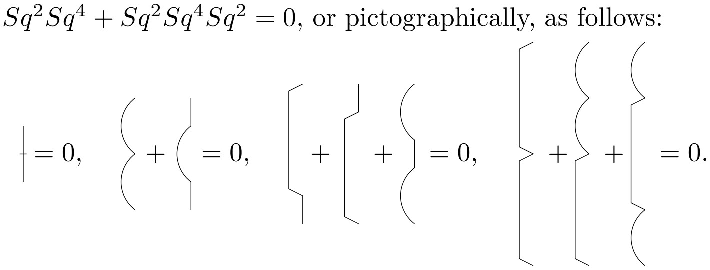
Douglas- Henriques-Hill
0810.2131

Appendix. The subalgebra $\mathcal{A}(2)$ of the Steenrod algebra

Here we include a portrait of the subalgebra of the 2-primary Steenrod algebra generated by Sq^1, Sq^2, and Sq^4.



The relations are $Sq^1Sq^1 = 0$, $Sq^2Sq^2 + Sq^1Sq^2Sq^1 = 0$, $Sq^4Sq^1 + Sq^1Sq^4 + Sq^2Sq^1Sq^2 = 0$, and $Sq^4Sq^4 + Sq^2Sq^2Sq^4 + Sq^2Sq^4Sq^2 = 0$, or pictographically, as follows:



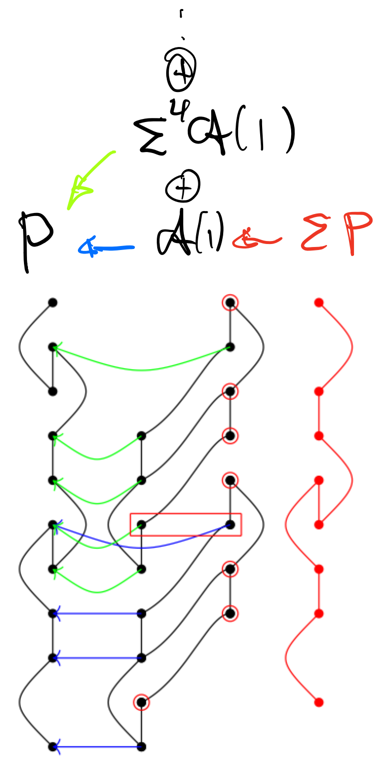


FIGURE 5.8. The $\mathcal{A}(1)$-resolution of P

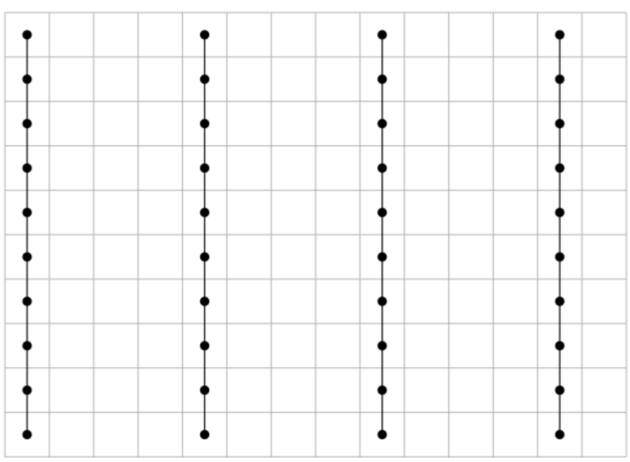


FIGURE 5.9. $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(P, \mathbf{F}_2)$

6 Computing ko via Adams

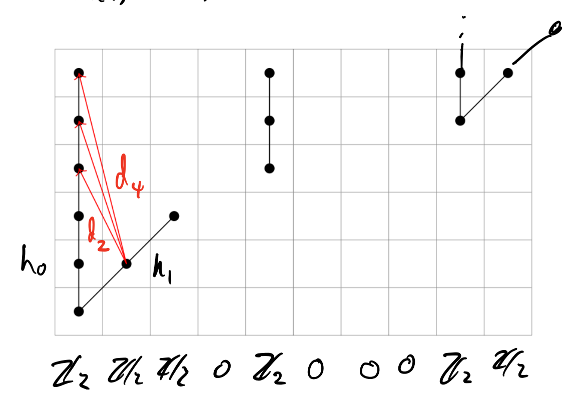
<u>Fact</u>: For X spectrum, $H^*(ko \wedge X) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} H^*(X)$

So $\underbrace{\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X), \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(ko \wedge X), \mathbb{F}_2)}_{\text{change of rings}}$

$$\Downarrow$$

$$\pi_{t-s}(ko \wedge X) = ko_{t-s}(X)$$

$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow ko_{t-s}$



Can show d_r are zero:

$0 = d_r(h_0 h_1)$

$= \cancel{d_r h_0 h_1} + h_0 d_r h_1$

If $d_r h_1 = h_0^{r+1}$, then $h_0^{r+2} = 0$ ↯

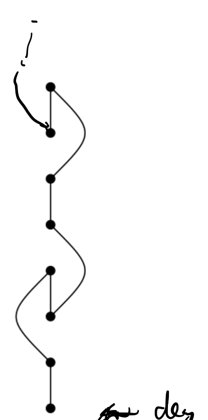
$\Rightarrow d_r = 0 \quad \forall r$

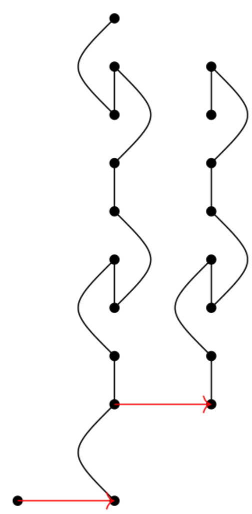
Suppose we want to compute $ko_*(\mathbb{R}P^\infty)$.

We already know $H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[x]$

as an $\mathcal{A}(1)$-module:

$Sq^1(x^n) = n x^{n+1} \qquad Sq^2(x^n) = \binom{n}{2} x^{n+2}$





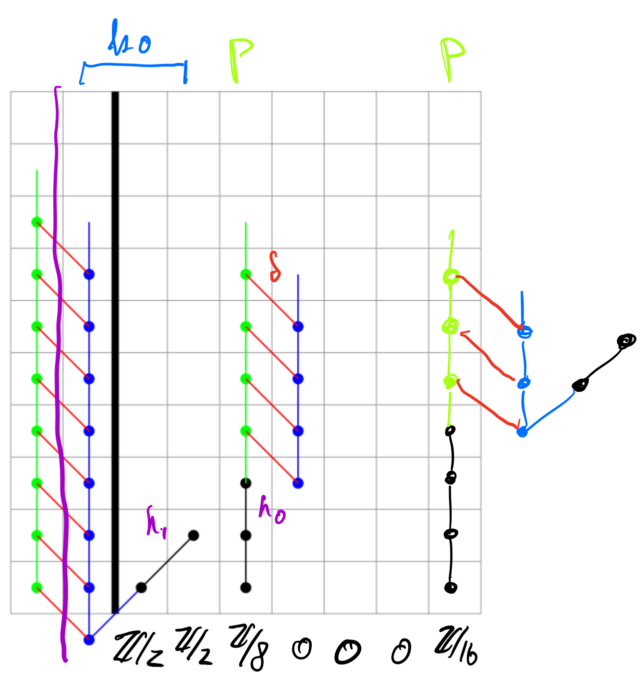
$\Rightarrow$

$$\begin{array}{c}
\vdots \\
\downarrow \\
\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\Sigma H^*(\mathbb{R}P^\infty), \mathbb{F}_2) \\
\downarrow \\
\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(P, \mathbb{F}_2) \\
\downarrow \\
\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \\
\downarrow \delta \\
\mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)}(\Sigma H^*(\mathbb{R}P^\infty), \mathbb{F}_2) \\
\downarrow \\
\vdots
\end{array}$$

$0 \to \mathbb{F}_2 \to P \to \Sigma H^*(\mathbb{R}P^\infty) \to 0$

δ has Adams degree $(-1,1)$

Put the two Ext's in the same Adams chart



$\delta \neq 0$ if it can be by exactness

$$\boxed{ko_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2}$$
$$ko_2(\mathbb{RP}^\infty) = \mathbb{Z}/2$$
$$ko_3(\mathbb{RP}^\infty) = \mathbb{Z}/8$$
$$ko_4(\mathbb{RP}^\infty) \cong 0$$
$$ko_5(\mathbb{RP}^\infty) \cong 0$$
$$ko_6(\mathbb{RP}^\infty) \cong 0$$
$$ko_7(\mathbb{RP}^\infty) \cong \mathbb{Z}/16$$

<u>Sidenote</u>: using that in low degrees $MSpin$ agrees with ko, can show that these are equal to $\Omega^{Pin^-}_{n-1}$ (Freed-Hopkins)

7 The dual Steenrod algebra & the cobar complex

$\mathcal{A}$ is a cocommutative Hopf algebra with simple coproduct

$$\Delta(Sq^i) = \sum_{a=0}^{i} Sq^a \otimes Sq^{i-a}$$ and hard product.

The dual Steenrod algebra is $\mathcal{A}_* := \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}, \mathbb{F}_2)$ linear dual

$\cong \pi(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$

It is a

commutative Hopf algebra with simple product and reasonable coproduct

It has the Milnor basis $\{\xi_{i_1} \cdots \xi_{i_k} : i_j > 0, k \geq 0\}$

where ξ_i is dual to $Sq^{2^{i-1}} Sq^{2^{i-2}} \cdots Sq^2 Sq^1$.

In fact $\mathcal{A}_*$ is polynomial in ξ_i. The reasonable coproduct is

$$\Delta \xi_i = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k \qquad (\xi_0 := 1)$$

The primitives are $\xi_1^{2^n}$ $\Leftrightarrow$ indecomposables in $\mathcal{A}$

$\Leftrightarrow h_n \in \mathrm{Ext}^{1,2^n}(\mathbb{F}_2, \mathbb{F}_2)$

$\Delta x = 1 \otimes x + x \otimes 1$

$\mathcal{A}_*(n)$ is a quotient of $\mathcal{A}_*$ by $(\xi_1^{2^n}, \xi_2^{2^{n-1}}, \ldots, \xi_{n-1}^2, \xi_n, \xi_{n+1}, \ldots)$

$H_*(X)$ is an $\mathcal{A}_*$ comodule

Prop The E_2-term of the homology Adams spectral sequence is the cohomology of the cobar complex

$$H_*(X) \to \ker\varepsilon \otimes H_*(X) \to \ker\varepsilon \otimes \ker\varepsilon \otimes H_*(X) \to \cdots$$

with differential

$$d(a_1|\dots|a_s|x) = 1|a_1|\dots|a_s|x +$$
$$\sum_{i=1}^{s} a_1|\dots|a_{i-1}|a_i'|a_i''|a_{i+1}|\dots|a_s|x$$
$$+ a_1|\dots|a_s|x'|x''$$

where $\Delta a_i = a_i' \otimes a_i''$ and $x \mapsto x' \otimes x''$ comodule structure