Sym mon ∞-cats

Commutative algebra objects in sym mon cat

$(Ab, \underset{\mathbb{Z}}{\otimes}) \overset{CAlg(-)}{\rightsquigarrow}$ Commutative Rings

only commutative

E_∞-algebras in sym mon ∞-cat

$(Sp, \wedge) \overset{CAlg(-)}{\rightsquigarrow} E_\infty$-ring spectra

Sym mon 1-cats recall

A lax sym mon functor $F: \mathcal{C} \to \mathcal{D}$ has

natural transformations $F(c) \otimes F(c') \to F(c \otimes c')$, $\mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$

Compatibility with α_{assoc} & β_{braid}

Ex an algebra in $\mathcal{C}$ is a lax mon functor $* \to \mathcal{C}$ terminal (unique sym mon str on a pt)

Need to encode ∞ many compatibilities such as pentagon "coherent homotopy"

Def $Fin_* := \{$finite pointed sets$\} \ni \{*, 1, \ldots, n\} =: \langle n \rangle$

[combinatorics machine Γ^{op}

$m_n : \langle n \rangle \to \langle 1 \rangle \qquad m_n(i) = 1 \quad \forall i \neq *$ "operadic stuff"

$g_i : \langle n \rangle \to \langle 1 \rangle$ inert $\leftarrow$ throw shit away

$j \longmapsto \begin{cases} 1 & j = i \\ * & j \neq i \end{cases} \qquad |\alpha^{-1}(i)| = 1 \quad \forall i \neq *$

Segal's special Γ-spaces:

<u>Def'n</u> A <u>sym mon ∞-cat</u> is a ∞-functor (∞,1)-cat

$$\underline{C} : Fin_* \to Cat_\infty$$ satisfying the <u>Segal condition</u>:

s.t. $\forall n \geq 0$ $\underline{C}(\langle n \rangle) \xrightarrow{(\rho^i)} \prod_{i=1}^{n} \underline{C}(\langle 1 \rangle)$ equivalence

e.g. $n=0 \Rightarrow \underline{C}(\langle 0 \rangle) = *$

choice of inverse needed for def'n of $\otimes$.

$$C := \underline{C}(\langle 1 \rangle) \qquad \otimes : C \times C \xleftarrow[\sim]{(\rho_1,\rho_2)} \underline{C}(\langle 2 \rangle) \xrightarrow{m_2} \underline{C}(\langle 1 \rangle)$$

$$\mathbb{1} : * = \underline{C}(\langle 0 \rangle) \to \underline{C}(\langle 1 \rangle) = C$$

$\langle 3 \rangle$

$$\begin{tikzcd}[row sep=small]
3 \arrow[dr] & & \\
2 \arrow[r] & 2 \arrow[dr] & \\
1 \arrow[r] & 1 \arrow[r] & 1 \\
* & * & *
\end{tikzcd}$$

$$C^{\times 3} \to C$$

$$c_1, c_2, c_3 \mapsto c_1 \otimes (c_2 \otimes c_3)$$

unitor, braiding

with courage can derive pentagon (which is a nat. modification)

<u>Claim</u> Sym mon 1-cat $\overset{\text{Nerve}}{\Longrightarrow}$ Sym mon ∞-cat

$$Fin_* \to Cat_\infty$$
$$\langle n \rangle \mapsto N(C)^{\times n}$$

def'n A strong sym mon functor is a nat transform $\underline{\mathcal{C}} \Rightarrow \underline{\mathcal{D}}$

$\Rightarrow$ $\mathrm{SymMonCat}_\infty \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_\infty)$ full subcat

Problem We are considering Cat_∞ as a $(\infty,1)$ cat, not $(\infty,2)$-cat.

$(\infty,2)$ is necessary for lax sym mon functor

Prop $\mathcal{C}$ sym mon $\Rightarrow$ $h\mathcal{C}$ sym mon (converse false)

Remark Fin_* and Δ^{op} for monoidal

Fin_*: E_∞ $\quad$ Δ^{op}: A_∞ $\quad$ determines combinatorics

Grothendieck op-fibrations

first 1-categorical: $F : \mathcal{B} \to \mathrm{Cat}_1$ (bicat of 1-cats) 2-functor pseudo functor

Def the Grothendieck construction of F : $\mathcal{E} := \int F \in \mathrm{Cat}_1$ category of elements

has obj: (b, e) $b \in \mathrm{ob}\,\mathcal{B}$, $e \in \mathrm{ob}\, F(c)$

mor $(b_1, e) \to (b_2, e)$: $\alpha : b_1 \to b_2$ $\exists$: $F(\alpha)(b_1) \to b_2$

$\exists$ forget ful functor $\int F \xrightarrow{p} \mathcal{B}$ $(b, x) \mapsto b$ $p^{-1}(b) = F(b)$

associated grothendieck fibration

Th $\mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_1) \to \mathrm{Cat}_1/\mathcal{B}$

$F \mapsto (\int F \to \mathcal{B})$ fibration

is faithful onto the (non-full) sub-category of $\mathrm{Cat}_1/\mathcal{C}$ with Grothendieck/co Cartesian op fibrations as objects (fibered categories), "cocartesian functors" as 1-morphisms ($\mathcal{E}$ takes cocartesian arrows to cocartesian arrows) & "vertical natural transformations" as 2-morphisms

coExample stacks $\mathrm{Man} \to \mathrm{Cat}_1$, $M \mapsto \mathrm{Vect}\, M$ (not op)

Example G group discrete, $F : */\!/G \to 1\text{-}\mathrm{Cat}$ any 2-functor $\mathcal{C} := F(*)$ (1-cat with 1 object & G as morphisms) "action $G \curvearrowright \mathcal{C}$"

$\mathcal{B}$

$\mathrm{ob} \int F = \mathrm{ob}\, \mathcal{C}$

$\mathrm{mor}_{\int F}(d_1, d_2) = \{ \alpha = g \in G,\ F(g)[d_1] \to d_2 \}$

composition uses $F(g_1) \circ F(g_2) \simeq F(g_1 g_2)$ "homology quotient"

<u>subexample</u> $\mathcal{C} = */\!/H \Rightarrow F$ "nonabelian cocycle"

& $1 \to H \to \int F \to G \to 1$ classifies extensions

fiber

forgetful functor

$$\begin{tikzcd}
e_1 \arrow[r, "f"] \arrow[d, Rightarrow] \arrow[rr, bend right, "\forall"'] & e_2 \arrow[r, dashed, "\exists!"] \arrow[d, Rightarrow] & e_3 \arrow[d] \\
b_1 \arrow[r] \arrow[rr, bend right] & b_2 \arrow[r, "\forall"] & b_3
\end{tikzcd}$$

f cocartesian

$$\begin{tikzcd}
c_1 \arrow[r, dashed, "\exists"] \arrow[d, Rightarrow] & \exists c_2 \arrow[d, Rightarrow] \\
b_1 \arrow[r] & b_2
\end{tikzcd}$$

cocartesian ← in some sense unique

opfibration

$\Rightarrow$ covariant transport $f^{-1}(b_1) \to f^{-1}(b_2)$

$(\mathcal{C}, \otimes)$ sym mon 1-cat $\Rightarrow F: \mathrm{Fin}_* \to \mathrm{Cat}_\infty$

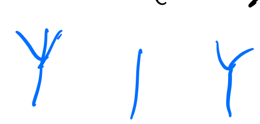
the <u>category of operators</u> is $\int F$:

<u>obj</u>: possibly empty finite ordered lists $c_1, \dots, c_n \in \mathcal{C}$

<u>mor</u> $((c_1, \dots, c_n), (d_1, \dots, d_m))$

consists of $\alpha: \langle n \rangle \to \langle m \rangle \in \mathrm{Fin}_*$ & $\forall k \in \langle m \rangle \setminus \{*\}$

a map $\bigotimes_{i \in \alpha^{-1}(\langle k \rangle)} c_i \to d_k$



K=1 K=2 K=3 K=4

$c_1 \otimes c_2 \otimes c_3 := (c_1 \otimes c_2) \otimes c_3$

obvious composition

forget fnl

$\mathcal{C}^\otimes \xrightarrow{p} \mathrm{Fin}_*$

Def $p: E \to B \in$ inner sSets

An *edge* $f: e_1 \to e_2$ of E is *p-cocartesian* if every lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, "d"'] \arrow[dr, bend left, "f"] & \\ \Lambda^n_0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a sol'n $\forall n \geq 2$

several equivalent def'ns

p is *coCartesian fibration* if

(i) p inner fibration (Fibs in Joyal model str) — fibrant = ∞-cat

(ii) $\forall$ edge $\alpha: b_1 \to b_2$ in S $\forall e_1 \in p^{-1}(b_1)$ vertex

$\exists$ p-coCartesian edge $f: e_1 \to e_2$ s.t. $p(f) = \alpha$

The fibers

$$\begin{tikzcd} p^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d] \\ \Delta^0 \arrow[r] & B \end{tikzcd}$$

are ∞-cats

Prop For $\mathcal{E}, B$ ∞-cats, fibers

assemble into a functor $\alpha: b_1 \to b_2 \rightsquigarrow \alpha_!: p^{-1}(b_1) \to p^{-1}(b_2)$

"straightening" Grothendieck construction = "unstraightening"

Def'n of Lurie A *sym mon ∞-cat* is a cocartesian fibration

$\mathcal{C}^{\otimes} \to Fin_*$

$$\left(\begin{tikzcd} \mathcal{C}^{\otimes}_{\langle n\rangle} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\otimes} \arrow[d] \\ \Delta^0 \arrow[r, "\langle n\rangle"] & \mathrm{Fin}_* \end{tikzcd}\right.$$

$\& \ \exists$ lifts $\alpha_! : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle m\rangle} \quad \forall \alpha : \langle n\rangle \to \langle m\rangle \in \mathrm{Fin}_* \Big)$

& the maps $\mathcal{C}^{\otimes}_{\langle n\rangle} \xrightarrow{(\rho_i)_!} \mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1\rangle}$

are equivalences $\forall n \geq 0$

<u>Example</u> $\mathcal{C}$ with finite products & $\otimes = \times$

for experts

<u>Remark</u> N_Δ(nice simplicial model cat with sym $\otimes$) is sym mon

simplicial nerve

<u>Th</u> Sp & $D(R)$ admit unique str of closed

sym mon ∞-cat with units $\mathbb{S}$ & $R[0]$

$\otimes = \wedge$ & $\otimes = \otimes^{\mathbb{L}}_R$

$-\otimes c : \mathcal{C} \to \mathcal{C}$ admits a right adjoint $\underline{\mathrm{Hom}}(c,-)$

$(Ch^- A)[\text{quasi-isos}^{-1}] \simeq D^- A$ or dg-Nerve of fib-cofib objects

Def A lax sym mon functor is a functor + nat tr α

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr] & \arrow[d, phantom, "\Leftarrow \alpha"] & \mathcal{D}^{\otimes} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

s.t. F sends cocartesian lifts of inert morphisms to cocartesian edges.

strongly monoidal if all cocartesian edges preserved

Note α is unique if it exists, because of the condition

2. A lax sym mon functor of sym mon 1-cat induces one of ——— ∞-cats

Def An algebra in $\mathcal{C}^{\otimes}$ is a lax sym mon $* \to \mathcal{C}^{\otimes}$

category of operators of $\mathcal{C} = *$

$$\begin{tikzcd} \mathrm{Fin}_* \arrow[r, "s"] \arrow[dr, "id"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ & \mathrm{Fin}_* \end{tikzcd}$$

operadic section

Concretely: $A := s\langle 1\rangle \in \mathcal{C}$ "the actual algebra"

$s\langle 2\rangle = (A_1, A_2) \in \mathcal{C}\langle 2\rangle \simeq \mathcal{C} \times \mathcal{C}$

& $s\langle 2\rangle \xrightarrow{s(m_2)} s\langle 1\rangle$ gives $A_1 \otimes A_2 \to A$

$s(\rho_i)$ cocartesian $\Rightarrow$ supplies $A_1 \cong A_2 \cong A$

$s\langle 0\rangle \in \mathcal{C}\langle 0\rangle \to \mathcal{C}\langle 1\rangle$ gives unit

Commutativity, ...

Def A <u>E_∞-ring spectrum</u> is an algebra object in $(Sp, \wedge)$

Can now define modules, limits, spectrum objects in $\mathcal{C}$

Th $S \xrightarrow{(\cdot)_+} S_* \xrightarrow{\Sigma^\infty} Sp$ sym mon functor