Symmetry-Protected Topological Phases in the Bogoliubov-de Gennes Framework

1. SPTs on 1-particle Hilbert space
2. 2^{nd} Quantization & BdG Hamiltonians
3. BdG Hamiltonians on Nambu space
4. Fermionic Symmetries
5. Quasi-Particle vacua & K-theory

Bonus: Particle-hole symmetry

Free Fermion SPTs on 1-particle Hilbert space

G topological group $\qquad G \xrightarrow{\phi} \mathbb{Z}/2 = \{\pm 1\}$

$\rho: G \to U(V) \amalg AU(V)$ $\quad$ $\mathbb{C}$-Hilbert $\qquad \phi(g) = -1 \Leftrightarrow \rho(g)$ anti-unitary

a gapped G-symmetric Hamiltonian:

- $h: V \to V$ s.a.
- $[h, \rho(g)] = 0$
- $0 \notin \operatorname{spec} h$

$\rightsquigarrow$ Spectral Flattening $\quad (1-t)h + \dfrac{h}{|h|} t$

$\Rightarrow$ grading $\quad V = V_+ \oplus V_-$

$$\varepsilon = \frac{h}{|h|} \qquad \varepsilon = +1 \qquad \varepsilon = -1$$

<u>Karoubi</u>: $\varepsilon \approx$ generator of

$$K_0(C^*_{\mathbb{C}}(G)) =: SPT_G^{\text{charge-conserving}}$$

Goal Repeat for non-charge conserving Hamiltonians

2^{nd} Quantization

$$F = \Lambda^{\bullet} V = \bigoplus_{i \geq 0} \Lambda^i V$$

$\{e_i\}$ orthonormal basis of V

$\Rightarrow a_i^\dagger = e_i \wedge -$ $\quad a_i = \iota_{e_i}$

$H := [\hat{h}, -] : W \to W$

$\quad\quad \overline{\mathrm{Span}_{\mathbb{C}}\{a_i, a_i^\dagger\}}$

2^{nd} Q. hamiltonian $\hat{h} = \sum h_{ij} (a_i^\dagger a_j - a_i a_j^\dagger)$

don't allow here $\Rightarrow$ ~~interactions: $a_i^\dagger a_j^\dagger a_k a_l$~~ $\quad E_i a_i^\dagger a_i$

not charge conserving: $a_i a_j, \; a_i^\dagger a_j^\dagger$

$$\widehat{e^{ith}} (v_1 \wedge \ldots \wedge v_n) = e^{ith} v_1 \wedge \ldots \wedge e^{ith} v_n$$

$$W = V \oplus V^* = \overline{\mathrm{Sp}_{\mathbb{C}}\{a_i, a_i^\dagger\}}$$

$$H = \begin{pmatrix} h & \Delta \circ c^{-1} \\ -c \circ \Delta & -c \circ h \circ c^{-1} \end{pmatrix} \begin{array}{c} V \\ V^* \end{array}$$

$c : V \to V^*$ antilinear iso

$\Delta^\dagger = -\Delta : V \to V$ antilinear

BdG Hamiltonians act on Nambu space

Idea: forget the structure of $V \subseteq V \oplus V^*$

Def A Nambu space is the complexification

$W = \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$

of a real Hilbert space $\mathcal{M}$ $\Rightarrow W$ is $\mathbb{C}$ Hilb

$\gamma(m \otimes z) = m \otimes \bar{z}$ real structure

Def A BdG-Hamiltonian is s.a. $H : W \to W$ imaginary

i.e. $-H\gamma = \gamma H$

iH skewadjoint on $\mathcal{M}$

Fermionic Symmetry Groups

Def A <u>fermionic group</u> $(G, \theta, (-1)^F)$ consists of

- G topological group
- $\theta : G \to \{\pm 1\}$ continuous homomorphism
- $(-1)^F \in G$ central, $((-1)^F)^2 = 1$, $\theta((-1)^F) = 1$

<u>Example</u> (class DIII) $G = \{1, (-1)^F, T, (-1)^F T\} \simeq \mathbb{Z}/4$

$$T^2 = (-1)^F$$

$$\theta(T) = -1$$

Def A (free fermionic) <u>representation of $(G, \theta, (-1)^F)$ on (W, γ)</u>

is $R : G \to U(W) \amalg AU(W)$ homom.

- $R(g)$ is (anti)-linear depending on $\theta(g) = -1$
- $R((-1)^F) = -1$
- $R(g)\gamma = \gamma R(g)$

Example (class A) $\quad G = \{1, (-1)^F, iQ, (-1)^F iQ\} \cong \mathbb{Z}/4$

$\Theta = 1 \qquad (iQ)^2 = (-1)^F$

$\Rightarrow W = V \oplus \gamma(V)$

$V = \{Q=+1\} \oplus \{Q=-1\}$

$$H = \begin{pmatrix} h & \Delta \circ c^{-1} \\ -c\circ\Delta & -c\circ h\circ c^{-1} \end{pmatrix} \qquad \begin{matrix} \Delta^\dagger = -\Delta \\ h^\dagger = h \end{matrix}$$

$[H, iQ] = 0 \iff \Delta = 0$

Quasi-Particle Vacua & K-theory

H gapped BdG Hamiltonian

$W_+ :=$ positive energy fields $\subseteq W$

$\big(=$ spectral projection for $(0,\infty) \big)$

$\Rightarrow$ Flattened Hamiltonian $\frac{H}{|H|}$

$\Leftrightarrow$ "quasi-particle vacuum" $J = i\frac{H}{|H|}$

Def A G-invariant quasi-particle vacuum is $J \in O(W)$ real $\mathbb{C}$-str s.t.

$$J R(g) = \theta(g) R(g) J$$

For $C^*_f(G)$ " $= \frac{C^*_{\mathbb{R}}(G)}{(1+(-1)^F)}$ " $\mathbb{Z}/2$-graded by θ

Th $\{$reps of $(G, (-1)^F, \theta)\}$

G-invariant QPVs

$\Updownarrow$

$$\left\{\text{reps of } Cl_{+2} \otimes_{\mathbb{R}} C^*_f(G)^{op}\right\} \quad \check{\mathbb{Z}/2}\text{-gradings}$$

$e_1 = \gamma$ $\qquad e_1 e_2 = i$

$$\Rightarrow SPT^{free}_{(G,(-1)^F,\theta)} = KO_2(C^*_f(G))$$

$$G_b := G/\langle(-1)^F\rangle$$

$V \mapsto \bar{V}$

$$Aut_{Vect_{\mathbb{R}}}(SVect_{\mathbb{C}}) \simeq \mathbb{Z}/2 \times B\mathbb{Z}/2$$

θ $\qquad G_b$ $\qquad V \mapsto (-1)^{1.1} \in Aut\, V$