# Operator algebra course

Luuk Stehouwer

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## Exercise sheet 1

Solutions to the following 6 questions are to be handed in **September 18 at the beginning of class**. Writing the solutions in LaTeX is preferred but not required.

*Exercise* 1. Let $A$ be a Banach algebra and $a \in A$. Show that left multiplication by $a$ defines a bounded operator $A \to A$.

*Exercise* 2. Let $A$ be a $C^*$-algebra. We abuse notation and denote the scalar multiple of the unit $1 \in A$ by $\lambda \in \mathbb{C}$ again by $\lambda \in A$.

1. Show that $\|1\| = 1 \in \mathbb{C}$ if and only if $A \neq \{0\}$.[1]

2. Show that we do always have that $\|\lambda\| = |\lambda|$ for $\lambda \in \mathbb{C}$ if we interpret both sides as being in $A$.

3. Give an example of a nonzero Banach algebra for which $\|1\| \neq 1$.

*Exercise* 3. The purpose of this exercise is to show that for very bad compact spaces $X, X'$, we can have $C(X) \cong C(X')$ even when $X \ncong X'$.

Let $S$ be a set.

1. Define the *cofinite topology* on $S$ by those open subsets $U$ such that either $U = \emptyset$ or $S \setminus U$ is finite. Show that this defines a topology on $S$.

2. Show that $S$ is compact.

3. Suppose that $S$ is finite. Prove that the $C^*$-algebras $C(S)$ and $\ell^\infty(S)$ are isomorphic.

4. Suppose that $S$ is infinite. Show that the $C^*$-algebra $C(S)$ is isomorphic to $\mathbb{C}$. (Hint: to show that every continuous map to $\mathbb{C}$ is constant, pick $z_1, z_2 \in \mathbb{C}$ distinct and look at the preimage of nonintersecting open disks around them)

*Exercise* 4. The purpose of this exercise is to compare the Hilbert direct sum with the algebraic direct sum of vector spaces. If $\{V_i\}_{i \in I}$ is a collection of vector spaces, the algebraic direct sum is defined as

$$\bigoplus_{i \in I}^{alg} V_i = \{(v_i)_{i \in I} : v_i \neq 0 \text{ for only finitely many } i\}.$$

Let $\{\mathcal{H}_i\}_{i \in I}$ be a family of Hilbert spaces with $\mathcal{H}_i \neq \{0\}$ for all $i \in I$.

[1] Note that the set $\{0\}$ has a unique $C^*$-algebra structure.

1. Show that $\bigoplus_{i\in I}^{alg} \mathcal{H}_i$ is a linear subspace of the Hilbert space direct sum $\bigoplus_{i\in I} \mathcal{H}_i$. Explain why this fact implies $\bigoplus_{i\in I}^{alg} \mathcal{H}_i$ is an inner product space.

2. Show that $\bigoplus_{i\in I}^{alg} \mathcal{H}_i$ is a proper subset of $\bigoplus_{i\in I} \mathcal{H}_i$ if and only if $I$ is an infinite set.

3. Show that $\bigoplus_{i\in I}^{alg} \mathcal{H}_i$ is a Hilbert space if and only if $I$ is a finite set.

*Exercise* 5. Let $A$ be a $C^*$-algebra.

1. Let $a, b \in A$ be self adjoint. Show that $ab$ is self-adjoint if and only if $a$ and $b$ commute.

2. Show that if $a \in A$ is self-adjoint, then $\|a\|^2 = \|a^2\|$.

3. Find an example of a $C^*$-algebra $A$ and an element $a \in A$ such that $\|a\|^2 \neq \|a^2\|$.

*Exercise* 6. Let $\{A_i\}_{i\in I}$ be a family of $C^*$-algebras. Show that $\bigoplus_{i\in I} A_i$ can be realized as the product of $\{A_i\}_{i\in I}$ in the category of $C^*$-algebras. (Hint: you may use the fact we will prove later in the lectures that every $*$-homomorphism $\phi\colon A \to B$ into a nonzero $C^*$-algebra $B$ is bounded of operator norm 1.)

## Bonus exercises

Here are some extra exercises to improve your general understanding of the material in class. These exercises are not to be included in the hand in assignment. Some exercises have the added purpose to reply to certain questions by students during and after class.

*Exercise* 7. Show that a subset $U \subseteq X$ of a topological space is open if and only if every $x \in U$ has an open neighborhood again contained in $U$.

*Exercise* 8. Let $f, g\colon A \to B$ be $*$-homomorphisms between $C^*$-algebras $A, B$.

1. Show that $\{a \in A : f(a) = g(a)\}$ is a $C^*$-subalgebra of $A$.

2. Show that this $C^*$-subalgebra is the equalizer in the category of $C^*$-algebras.

*Exercise* 9. Let $A$ be a $C^*$-algebra and let $a \in A$. Define the *real part of* $a$ to be $\Re a := \frac{a+a^*}{2}$ and the *imaginary part of* $a$ to be $\Im a = \frac{a-a^*}{2i}$

1. Show that $\Re(a^*) = \Re(a)$ and $\Im(a^*) = -\Im(a)$.

2. Show that $a$ is self-adjoint if and only if $\Re(a) = a$.

3. Find an example of a $C^*$-algebra $A$ and an element $a \in A$ such that $\Re(a)$ and $\Im(a)$ don't commute.

*Exercise* 10. We made a big fuss about whether linear maps between Banach spaces are bounded (continuous) or not. The purpose of this exercise is to show that in finite dimensions, every linear map is bounded.

1. Show that
$$\|x\| = \sum_{i=1}^{n} |x_i|$$
defines a norm on $\mathbb{C}^n$.

2. Let $T\colon \mathbb{C}^n \to \mathbb{C}^m$ be a linear map, where we equip $\mathbb{C}^n$ and $\mathbb{C}^m$ with the norm from part 1. Show that $\|Tx\| \leq \|x\| \max_i \|T(e_i)\|$ where $e_i$ is the standard basis.

3. Show that any map $T\colon X \to Y$ between finite-dimensional normed spaces is bounded. You can use the fact that if $\|.\|$ and $\|.\|'$ are two norms on the same finite-dimensional vector space $X$, then there exist constants $C_1, C_2 > 0$ such that $C_1\|x\| \leq \|x\|' \leq C_2\|x\|$ for all $x \in X$.

*Exercise* 11. Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of nonzero Banach algebras. Then

$$A = \bigoplus_{n\in\mathbb{N}} A_n := \{a_n \in A_n : \sup_n \|a_n\| < \infty\}$$

is a Banach algebra (you don't have to show this). Let $A_0 \subseteq A$ be the subset of those $a_n \in A$ such that $\lim_{n\to\infty} \|a_n\| = 0$.

1. Show that $A_0$ is a closed subspace of $A$.

2. Why is $A_0$ not a Banach algebra?

3. Show that $A_0$ is an ideal in $A$.

4. Prove or disprove: $A_0$ is a maximal ideal.

*Exercise* 12. The goal of this exercise is to show that there is no Hilbert space with a countable Hamel basis. Let $\mathcal{H}$ be a Hilbert space. We say that $v, w \in \mathcal{H}$ are *orthogonal* if $\langle v, w\rangle = 0$. A collection $\{e_i\}_{i\in I}$ is *orthonormal* if $\|e_i\| = 1$ for all $i \in I$ and the $e_i$ are pairwise orthogonal.

1. Prove that
$$\|\sum_{i=1}^n \lambda_i e_i\| = \sqrt{\sum_{i=1}^n |\lambda_i|^2}$$
if $\{e_1, \ldots, e_n\}$ is an orthonormal set.

2. Prove that an orthonormal set is linearly independent.

3. Suppose that $\{e_n\}_{n\in\mathbb{N}}$ is a countably infinite orthonormal set. Show that
$$\left(\sum_{n=1}^k \frac{e_n}{2^n}\right)_k$$
is a Cauchy sequence.

4. Prove that the limit of this sequence can't be written as a finite linear combination of elements of $\{e_n\}_{n\in\mathbb{N}}$.

*Exercise* 13. If $X$ is a Banach space, let $X^* = B(X, \mathbb{C})$ be the Banach space of bounded linear functionals. The purpose of this question is to give an example of a Banach space $X$ such that $X^* \not\cong X$.

Recall that the *closure* $\overline{A}$ of a subset $A \subset X$ of a topological space is the intersection of all closed sets in $X$ that contain $A$. The subset $A$ is called *dense* if $\overline{A} = X$. We say that a topological space is *separable* if it admits a countable dense subset.

Let $c_{00} \subseteq \ell^\infty := \ell^\infty(\mathbb{N})$ be the subset of sequences $x_n$ which are nonzero for finitely many $n$.

1. Show that $c_{00}$ is dense in $\ell^1 := \ell^1(\mathbb{N})$. Conclude that $\ell^1$ is separable.
2. Let $c_0$ be the subset of $\ell^\infty$ of sequences $x_n$ such that $\lim_n x_n = 0$. Why does it follow from Exercise 11 that $c_0$ is closed in $\ell^\infty$? Why does this imply that $c_{00}$ can't be dense in $\ell^\infty$?
3. For each subset $I$ of the positive integers $\mathbb{N}$, define $e_I \in \ell^\infty$ by

$$(e_I)_i = \begin{cases} 1, & i \in I \\ 0, & i \notin I. \end{cases}$$

Show that the balls $B_{\frac{1}{2}}(e_I)$ of radius $1/2$ with center $e_I$ for $I \subset \mathbb{N}$ form an uncountably infinite collection of disjoint open sets in $\ell^\infty$. Conclude that $\ell^\infty$ is not separable.
4. Show that if $X$ and $Y$ are homeomorphic topological spaces and $X$ is separable, then $Y$ is separable.
5. Define a map $\ell^\infty \to (\ell^1)^*$ by sending $x_n \in \ell^\infty$ to the functional

$$a_n \mapsto \sum_{n \in \mathbb{N}} x_n a_n.$$

Show that this is a well-defined bounded isomorphism of Banach spaces $\ell^\infty \cong (\ell^1)^*$.
6. Show that $(\ell^1)^* \ncong \ell^1$.

*Exercise* 14. The goal of this exercise to give an example of Banach spaces $X, Y$ such that $X \cong Y$ as vector spaces but $X \ncong Y$ as Banach spaces.

1. Using the notation of Exercise 13, define a map $\ell^1 \to (c_0)^*$ by sending $x_n \in \ell^1$ to the functional

$$a_n \mapsto \sum_{n \in \mathbb{N}} x_n a_n.$$

Show that this is a well-defined bounded isomorphism of Banach spaces $\ell^1 \cong (c_0)^*$.
2. Show that a Hamel basis of a separable infinite-dimensional Banach space has the cardinality of the continuum. You are allowed to use the continuum hypothesis and the result that there is no Banach space with countable Hamel basis (i.e. the generalization of Exercise 12 to arbitrary Banach spaces also holds). Conclude that $c_0$ and $\ell^1$ are isomorphic as vector spaces.
3. Show that $c_0$ and $\ell^1$ are not isomorphic as Banach spaces. (Hint: if they were, then $c_0^* \cong (\ell^1)^*$ as Banach spaces. You can use the results of Exercise 13)

*Exercise* 15. This exercise discusses reflexive Banach spaces. Let ev be the map $X \to X^{**}$ defined by $x \mapsto \mathrm{ev}_x$ where $\mathrm{ev}_x(f) = f(x)$ for $f \in X^*$.

1. Show that ev is a bounded linear map.
2. A Banach space is called *reflexive* if ev is an isomorphism of Banach spaces. Recall that the Riesz-Fréchet theorem gives an isomorphism $RF \colon \mathcal{H} \to \mathcal{H}^*$ of Banach spaces. Equip $\mathcal{H}^*$ with the unique Hilbert space structure making $RF$ into a unitary operator. Apply the Riesz-Fréchet theorem to $\mathcal{H}^*$ to show that every Hilbert space is reflexive.
3. Show that $c_0$ is not reflexive.