

Operator algebra course

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Exercise sheet 2

Solutions to the following 5 questions are to be handed in **September 25 at the beginning of class**.

Exercise 1. Let A be an algebra. Suppose $a \in A$ is invertible. Show that $\text{Spec}(a^{-1}) = (\text{Spec } a)^{-1}$. (Hint: $a^{-1} - \lambda^{-1} = (\lambda - a)(\lambda a)^{-1}$.)

Exercise 2. Let A be a C^* -algebra. Show without using the spectral radius formula that $\text{Spec}(a^*) = \overline{\text{Spec}(a)}$.

Exercise 3. Let A be a Banach algebra.

- (a) Use the spectral radius formula to show that if a is nilpotent, then $\text{Spec } a = \{0\}$.
- (b) Show directly from the definition (without using the spectral radius formula) that if a is nilpotent, then $\text{Spec } a = \{0\}$. (Hint: recall the geometric series formula for elements of a Banach algebra. In the current exercise it need not be the case that $\|a\| < 1$, but maybe something similar works?)

Exercise 4. Let A be a C^* -algebra and let $a \in A$ be self adjoint.

- (a) Show that $a^2 + 1 = (a + i)(a - i)$ is invertible.
- (b) Show that a^2 is positive.

Exercise 5. In this exercise, we study the spectrum of diagonal operators on Hilbert spaces, i.e. certain elements of the C^* -algebra $B(\mathcal{H})$. Let e_n be the standard orthonormal basis¹ of the Hilbert space $\ell^2(\mathbb{N})$. More specifically, e_n is the sequence with all zeroes except 1 at the n th spot. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers.

- (a) Show that $e_n \mapsto \lambda_n e_n$ uniquely defines a bounded operator S on $\ell^2(\mathbb{N})$ if and only if the sequence λ_n is bounded.
- (b) Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator on some Hilbert space $\mathcal{H}$. Show that if λ is an eigenvalue of T , then $\lambda \in \text{Spec } T$. Conclude that $\lambda_n \in \text{Spec } S$ for all $n \in \mathbb{N}$.
- (c) Show that $\text{Spec } S$ is the closure of the subset $\{\lambda_n\} \subseteq \mathbb{C}$.

¹Recall that in a Hilbert space $\mathcal{H}$, when we say orthonormal basis, we mean a collection of mutually orthogonal vectors e_i such that the closure of the span of the e_i is all of $\mathcal{H}$. We don't want to use a Hamel basis, as that concept that is mostly useful for finite-dimensional vector spaces.

- (d) Show that S is self adjoint if and only if $\lambda_n \in \mathbb{R}$ for all n and unitary if and only if $\lambda_n \in U(1)$ where $U(1) \subseteq \mathbb{C}$ is the unit circle.
- (e) Show that S is positive if and only if $\lambda_n \geq 0$ for all n .

Bonus exercises

Exercise 6. The purpose of this exercise is to give an example of a subalgebra $B \subseteq A$ of a C^* -algebra that does not satisfy spectral permanence.

Let $\{e_n\}_{n \in \mathbb{Z}}$ be the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. Define $Ue_n := e_{n+1}$.

- (a) Show that U extends uniquely to a bounded operator on $\ell^2(\mathbb{Z})$.
- (b) Show that U is unitary.
- (c) Consider the subalgebra of $B(\mathcal{H})$ generated by the operator U , i.e. the subalgebra consisting of polynomials in U . Let $A \subseteq B(\mathcal{H})$ be the norm closure of this subalgebra. Show that A is a Banach algebra.
- (d) Let $\mathcal{H}_+ \subseteq \mathcal{H}$ be the closure of the span of all vectors e_n for $n > 0$. Show that for all $a \in A$ we have $a(\mathcal{H}_+) \subseteq \mathcal{H}_+$.
- (e) Show that $0 \notin \text{Spec}_{B(\mathcal{H})} U$ but $0 \in \text{Spec}_A U$.

Exercise 7. Let X be a compact space and $f_1, f_2 \in C(X)$.

- (a) Show that the f_i are self adjoint if and only if they take values in the real numbers.
- (b) Show that $f_1 \leq f_2$ if and only if $f_1(x) \leq f_2(x)$ for all $x \in X$.

Exercise 8. Let $a \in A$ and $z \in \mathbb{C}$. Show that $\text{Spec}(a + z) = \text{Spec}(a) + z$.

Exercise 9. Let A be the algebra of complex rational polynomials in one variable z . More precisely, these are expressions of the form p/q with $p, q \in \mathbb{C}[z]$ polynomials and $q \neq 0$. Show that there exists $a \in A$ with $\text{Spec } a = \emptyset$. Conclude that A does not admit the structure of a C^* -algebra.

Exercise 10.

- (a) Let A be a C^* -algebra and let $u \in A$ be unitary. Show that $a \mapsto uau^{-1}$ is a $*$ -homomorphism.
- (b) Give an example of a C^* -algebra A and an element $b \in A$ such that $a \mapsto bab^{-1}$ is not a $*$ -homomorphism.

Exercise 11. Let a be self adjoint and $\lambda \in \mathbb{R}$. Show using the spectral radius formula that $\text{Spec } a = \{\lambda\}$ if and only if $a = \lambda$.

Exercise 12. Let Ban be the category of Banach spaces and contractions, i.e. objects are Banach spaces and morphisms $X \rightarrow Y$ are bounded linear maps $T: X \rightarrow Y$ such that $T \leq 1$.

- (a) Show that this is a category.
- (b) Show that the product in this category is given by $X \oplus Y$ with the ℓ^∞ norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$.
- (c) Show that the coproduct in this category is given by $X \oplus Y$ with the ℓ^1 norm $\|(x, y)\| = \|x\| + \|y\|$.

References