Exercise sheet 3

Solutions to the following 4 questions are to be handed in **October 9 at the beginning of class**.

Exercise 1. Let $n > 0$ be an integer and let $A \subseteq M_{2n}(\mathbb{C})$ be the subset of matrices of the form

$$\begin{pmatrix} \lambda I & B \\ 0 & \lambda I \end{pmatrix}, \tag{1}$$

where $\lambda \in \mathbb{C}, B \in M_n(\mathbb{C})$ and the identity and zero matrices are also n by n. We equip it with the norm coming from the fact that $M_{2n}(\mathbb{C})$ is a C^*-algebra. Since it is a finite-dimensional subspace, it is automatically complete and so a Banach space. Hence it is a Banach algebra once we can show that it is a subalgebra.

(a) Show that A is a subalgebra.

(b) Show that elements of A with $\lambda = 0$ are all nilpotent.

(c) Show that the space Φ_A only has a single point ϕ.

(d) Show that the Gelfand transform $A \to C(\Phi_A)$ sends a matrix of the form (1) to the function which sends ϕ to λ.

(e) Explain why A does not admit the structure of a C^*-algebra.

Exercise 2. Let $y \in [0, 1]$ and define

$$I = \{f \in C([0, 1]) : f(x) = 0 \text{ in some neighborhood of } y\}.$$

In other words, $f \in I$ if and only if there exists a neighborhood $U \subseteq [0, 1]$ of y such that $f|_U = 0$.

(a) Show that I is an ideal of $C([0, 1])$.

(b) Show that I is not closed. Here we take the topology on $C([0, 1])$ induced by its C^*-algebra structure.

Exercise 3. Let R be a commutative ring. Define

$$V(S) := \{m \text{ maximal ideal} : S \subseteq m\} \subseteq \Phi_R$$

(a) Show that $V(S) = V(I)$ if I is the ideal generated by S.

So we might as well assume S is an ideal from now on. The *Zariski topology* on the maximal ideal spectrum of R is defined by setting the closed subsets of Φ_R to be exactly the $V(I)$ as $I \subseteq R$ ranges over the ideals. A fact that you don't have to show here is that this defines a topology, i.e. finite unions of closed sets are closed and arbitrary intersections of closed sets are closed.

Now we consider the case where $R = A$ is a commutative Banach algebra.

(b) Translate $V(S)$ to the algebra homomorphism picture and show it corresponds to

$$\{\phi \in \Phi_A : \hat{a}(\phi) = 0 \, \forall a \in S\}.$$

(c) Show that every Zariski-closed set is also weak* closed.

(d) Show that the identity map $(\Phi_A, \text{weak* topology}) \to (\Phi_A, \text{Zariski topology})$ is continuous.

Exercise 4. Let $D \subseteq \mathbb{C}$ be the open unit disk of radius 1 and let $\overline{D}$ be its closure. Let A be the set of ($\mathbb{C}$-valued) holomorphic functions[1] on D with the property that they extend to a continuous function on $\overline{D}$.

The goal of this exercise is to show that $\Phi_A = \overline{D}$. This is interesting because $A \neq C(\overline{D})$ (there are continuous functions which are not holomorphic), so this is a counterexample to Gelfand duality.

(a) Show that A is an algebra.

(b) Show that there is a well defined injective algebra homomorphism $\mu\colon A \to C(\overline{D})$.

We will now use the norm on A induced by the C^*-algebra $C(\overline{D})$ under μ to make A into a Banach algebra. For this we need to know that A is closed, which you can show in Exercise 6 in case you are looking for a distraction from your obligations.

(c) Show that the function $\zeta\colon \overline{D} \to \Phi_A$ given by sending $z \in \overline{D}$ to $f \mapsto f(z)$ makes sense and is injective (Hint: consider evaluating $\phi \in \Phi_A$ on the function $f(z) = z$ in A).

(d) Show that ζ is continuous.

We will now apply the result proven in Exercise 6 that polynomials are dense in A.

(e) Use the fact that the subalgebra of A consisting of polynomials in a single variable z are dense to show that ζ is surjective.

(f) Show that ζ is a homeomorphism (hint: every bijective continuous map between compact Hausdorff spaces is a homeomorphism).

Bonus exercises

Exercise 5. Let $\mathcal{H} = \ell^2(\mathbb{N})$ be the Hilbert space of square summable sequences. Recall that the inner product is given by

$$\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} \overline{x}_n y_n$$

Define a map $T\colon \mathcal{H} \to \mathcal{H}$ by $(x_1, x_2, x_3, \dots) \mapsto (0, x_1/2, x_2/4, x_3/8, \dots)$

(a) Show that T is a well-defined bounded linear operator

(b) Show that $\|T\| = 1/2$

(c) Show by induction that $T^n(x_1, x_2, \dots) = (0, \dots, 0, x_1/2^{n(n+1)/2}, \dots)$ where all entries to the right of x_1 come with strictly smaller coefficients. Conclude that $\|T\| = 1/2^{n(n+1)/2}$.

(d) Use the spectral radius formula to show that $\operatorname{Spec}(T) = \{0\}$.

Exercise 6. Recall the notation of Exercise 4.

[1] For this exercise, you can use the basic properties of holomorphic functions, as for example discussed in the lecture notes.

(a) Show that the uniform limit of holomorphic functions on D is again holomorphic and conclude that A is closed in $C(\overline{D})$. (hint: $f\colon U \to \mathbb{C}$ is holomorphic if and only if for every curve γ in U we have
$$\int_\gamma f = 0)$$

(b) Given $f \in A$ and $0 < r < 1$, define $f_r(z) = f(rz)$. Show that $f_r \in A$.

(c) Show that $f_r \to f$ in A as $r \to 1$.

(d) Let $p_n \in A$ be the monomial $p_n(z) = z^n$ and let $\mathcal{P} \subseteq A$ be the linear span of p_n, i.e. the subalgebra of polynomials. Show that $f_r \in \overline{P}$. (Hint: f_r is holomorphic on a disk of radius > 1. Therefore it has a power series expansion there which converges uniformly on compact sets)

(e) Show that $\mathcal{P}$ is dense in A.

Exercise 7. Let $A = M_n(\mathbb{C})$ for $n > 1$. Show that there are no algebra homomorphisms $\phi\colon A \to \mathbb{C}$. (hint: note that $\phi(a) = 0$ if a is nilpotent. Look for nilpotents that multiply to diagonal matrices)

Exercise 8. A linear map $T\colon X \to Y$ between Banach spaces is called *contractive* if $\|Tx\| \leq \|x\|$.

(a) Show that every contractive map is a bounded operator

A *Banach $*$-algebra* is a Banach algebra A equipped with an involution $*\colon A \to A$ satisfying
$$(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^* \quad a^{**} = a \quad (ab)^* = b^* a^* \quad \|a^*\| = \|a\|$$
for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$

(b) Which axiom is missing in being a Banach $*$-algebra when you compare them to the axioms for being a C^*-algebra? And which axiom of being a Banach $*$-algebra does not occur in the definition of C^*-algebra we have seen?

(c) Let A be a C^*-algebra. Use the C^*-identity to show that $\|a^*\| \geq \|a\|$. Show that every C^*-algebra is a Banach $*$-algebra.

Let
$$\begin{aligned} Rep(A) = \{\pi\colon A \to B(\mathcal{H}) : &\ \mathcal{H} \text{ is a Hilbert space and} \\ &\pi \text{ is a contractive algebra homomorphism preserving } *\}. \end{aligned}$$
Define $R(a) = \sup_{\pi \in Rep(A)} \|\pi(a)\|$, where we used the usual operator norm on $B(\mathcal{H})$.

(d) Show $R(\lambda a) = |\lambda| R(a)$ and the triangle inequality for the candidate norm R.

(e) Let $I = \{a \in A : \pi(a) = 0 \,\forall \pi \in Rep(A)\}$. Show that $I = \{a \in A : R(a) = 0\}$.

(f) Show that I is a closed ideal.

(g) Show that R induces a norm $\|.\|$ on A/I which satisfies the C^*-identity

(h) Show that this norm makes the map $A \to A/I$ into a contractive algebra homomorphism.

(i) Show that with this norm A/I becomes a C^*-algebra.

Exercise 9. Let M be a metric space which is compact as a topological space. The goal of this exercise is to show that $C(M)$ is countably generated as a Banach algebra.

1. Show that M is *separable*, i.e. that it has a countable dense subset. (Hint: for fixed $n \in \mathbb{N}$ consider the collection of balls $B(x, 1/n)$ for $x \in M$)
2. If x_n is a countable dense subset of M, show that $f_n(x) = d(x, x_n)$ is a continuous function on M
3. Use the Stone-Weierstrass theorem to show that the closure of the algebra generated by the f_n is all of $C(M)$.

References