Operator algebra course

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Exercise sheet 4

Solutions to the following 4 questions are to be handed in **October 30 at the beginning of class**.

Exercise 1. Let $A = C(X)$ be the continuous functions on a compact Hausdorff space X. Recall that in class we have defined a bijection $p \mapsto p^{-1}(\{1\})$ between the collection of projections in A and the collection of clopens in X.

Show that the partial order on clopens given by $\subseteq$ corresponds to the partial order on projections given by restricting the partial order we have studied on self adjoint elements ($a \leq b$ iff $b - a$ is positive).

Exercise 2. Show that $T \mapsto T^*$ is a WOT-continuous map $B(\mathcal{H}) \to B(\mathcal{H})$.

Exercise 3. Recall that we have shown in the lecture that the sequence $E_{1,n}$ in $B(\ell^2(\mathbb{N}))$ WOT-converges to 0.

(a) Show that the sequence $E_{n,1}$ in $B(\ell^2(\mathbb{N}))$ WOT-converges to 0.

(b) Show that the multiplication map $B(\ell^2(\mathbb{N})) \times B(\ell^2(\mathbb{N})) \to B(\ell^2(\mathbb{N}))$ is not WOT-continuous (Hint: compute $E_{1,n}E_{n,1}$).

Exercise 4. Let X be the Banach space of formal symbols $\sum_{v,w \in \mathcal{H} \setminus \{0\}} \lambda_{v,w} e_{v,w}$ with

$$\sum_{v,w \in \mathcal{H} \setminus \{0\}} |\lambda_{v,w}| \|v\| \|w\| < \infty$$

and norm

$$\left\| \sum_{v,w \in \mathcal{H} \setminus \{0\}} \lambda_{v,w} e_{v,w} \right\| = \sum_{v,w \in \mathcal{H} \setminus \{0\}} |\lambda_{v,w}| \|v\| \|w\|$$

as defined in class. Show that these formal sums are always countable in the sense that $\lambda_{v,w} = 0$ for all but countably many v, w.

Bonus exercises

Exercise 5. Let $\mathcal{H} = \ell^2(\mathbb{Z})$. Define the operator $(Ux)_n = x_{n+1}$.

(a) Show that U is a bounded linear operator.

(b) Show that U is unitary.

(c) Show that the sequence U^n converges to 0 in the weak operator topology.

(d) Show that U^n does not converge in the norm topology to 0.

(e) Conclude that U^n does not converge at all in the norm topology.

Exercise 6. Show that for a fixed $T' \in B(\mathcal{H})$, the multiplication map $T \mapsto T'T$ is WOT continuous.

Exercise 7. Given $v, w \in \mathcal{H}$ define $E_{v,w}(u) = \langle w, u \rangle v$. An operator is *finite rank* if it is in the subalgebra generated by the $E_{v,w}$ (we don't take any closure).

(a) Show that T commutes with $E_{v,w}$ if and only if there exists $\lambda \in \mathbb{C}$ such that v is an eigenvector of T with eigenvalue λ and w is an eigenvector of T^* with eigenvalue $\overline{\lambda}$. (Hint: $\lambda = \frac{\langle w, Tw \rangle}{w, w}$)

(b) Let $A \subseteq B(\mathcal{H})$ be the subalgebra of finite rank operators. Show that $A' = \mathbb{C} \subseteq B(\mathcal{H})$.

(c) Conclude that $B(\mathcal{H})' = \mathbb{C}$.