Operator algebra course

Luuk Stehouwer

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Exercise sheet 5

The solution to the following exercise has to be handed in **November 6 at the beginning of class**.

Exercise 1. Let $A_0 = \bigcup_n M_{2^n}(\mathbb{C})$, where we include $M_{2^n}(\mathbb{C})$ into $M_{2^{n+1}}(\mathbb{C})$ using

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Note that A_0 has the structure of a $*$-algebra induced by the C^*-algebra structure of $M_{2^n}(\mathbb{C})$. However, it is not itself a C^*-algebra, because it is not complete.

(a) Show that $\langle a, b\rangle = \frac{1}{2^n}\operatorname{Tr}(a^*b)$ defines an inner product on $M_{2^n}(\mathbb{C})$. (Hint: you can use that a^*a is positive)

(b) Show that the inclusion of $M_{2^n}(\mathbb{C})$ into $M_{2^{n+1}}(\mathbb{C})$ preserves the respective inner products. Conclude that we get an inner product on A_0. (Warning: we now have two norms on A_0; the one induced by the inner product and the C^*-norm)

(c) Given $a \in A_0$, show that the linear map $L_a \colon A_0 \to A_0$ given by left multiplication by a is bounded in the norm on A_0 induced by the inner product. (Hint: for matrices $a, b \in M_k(\mathbb{C})$, we have $\operatorname{Tr}(b^*a^*ab) \leq \|a\|^2 \operatorname{Tr}(b^*b)$, where $\|a\|$ is the operator norm)

(d) Let $(\mathcal{H}, \langle ., .\rangle_{\mathcal{H}})$ be the completion of A_0 with respect to the inner product, and let $\pi \colon A_0 \to \mathcal{H}$ denote the dense embedding. Show that the left multiplication map $a \mapsto L_a$ extends to a well-defined injective algebra homomorphism $L \colon A_0 \to B(\mathcal{H})$. (Hint: recall from lecture 4 that we can define bounded linear maps on dense subspaces)

(e) Show that $L_{a^*} \colon A_0 \to A_0$ is the adjoint of L_a with respect to $\langle -, -\rangle$. Conclude that $L \colon A_0 \to B(\mathcal{H})$ is a $*$-homomorphism.

(f) Let A be the von Neumann algebra $L(A_0)'' \subseteq B(\mathcal{H})$. Define the linear functional $\tau \colon A \to \mathbb{C}$ by $\tau(a) = \langle \pi(1), a\pi(1)\rangle_{\mathcal{H}}$. Show that τ is continuous in the weak operator topology and extends $a \mapsto \frac{1}{2^n}\operatorname{Tr}(a)$ from $M_{2^n}(\mathbb{C}) \subseteq A_0$ to A.

(g) Show that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. (Hint: for which topologies is $L(A_0)$ dense in A?)

We will now show that the *center*

$$Z(A) = \{a \in A : ab = ba \, \forall b \in A\}$$

of A is trivial, i.e. $Z(A) = \mathbb{C}$. This means that A is a *factor*.

(h) Given $a \in Z(A)$, define $\tau_a(b) = \tau(ab)$. Show that $\tau_a(b) = \tau(a)\tau(b)$. (Hint: you can use the fact that if $\tau' \colon M_k(\mathbb{C}) \to \mathbb{C}$ satisfies $\tau'(ab) = \tau'(ba)$, then $\tau'(a) = \frac{1}{k}\tau'(1)\,\mathrm{Tr}(a)$ for all $a \in M_k(\mathbb{C})$)

(i) Show that if $a \in Z(A)$, then $a = \tau(a)$, so that indeed $Z(A) = \mathbb{C}$.