Operator algebra course

Luuk Stehouwer

November 15, 2025

Exercise sheet 6

The solution to the following 4 exercises have to be handed in **November 25 at the beginning of class**.

Exercise 1. Let $(B, +, \cdot)$ be a Boolean algebra. Show that the sum is given by the symmetric difference formula $x + y = \sup\{x, y\} \cdot (x \cdot y)^\perp$

Exercise 2. Let S be a set and let $B \subseteq P(S)$ be the collection of subsets $T \subseteq S$ such that either T is countable or $S \setminus T$ is countable.

1. Show that B is a subring of the Boolean algebra $P(S)$ and so a Boolean algebra itself.
2. Show that B admits countable suprema.
3. Suppose S is uncountable. Show that B does not admit arbitrary suprema.

Exercise 3. Let A be a C^*-algebra.

1. Show that if $a \in A$ is self adjoint, then $e^{ia} \in A$ defined using functional calculus is unitary.
2. Let $u \in A$ be a unitary such that $\operatorname{Spec} u$ is not the whole unit circle (recall that the spectrum of a unitary is always a subset of the unit circle). Show that there exists a self adjoint element $a \in A$ such that $e^{ia} = u$. (Hint: can we stay on one branch of the complex logarithm?)

Exercise 4. In this exercise you will show that the Cantor space is Stone but not Stonean.

1. Let $\{X_i\}_{i \in I}$ be a family of Stone spaces. Show that $\prod_i X_i$ with the product topology is a Stone space.
2. Conclude that the Cantor space C, which is defined as the countable product $C = \prod_{n \in \mathbb{N}} \{0, 1\}$ of two element discrete spaces is Stone.

If you prefer, you can think of points in C as infinitely long binary numbers. By definition of the product topology, a basis of the topology is given by finite intersections of the opens of the form

$$\{(y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots) : y_1, \ldots, y_{k-1}, y_{k+1}, \cdots \in \{0, 1\}\}$$

for a fixed $x \in \{0, 1\}$.

3. Given a finite number of fixed binaries $x_1, \dots, x_k \in \{0,1\}$, show that

$$U_{x_1,\dots,x_k} := \{(x_1, x_2, \dots, x_k, y_{k+1}, y_{k+2}, \dots) : y_{k+1}, y_{k+2}, \dots \in \{0,1\}\} \subseteq C$$

is open.

4. Let $U \subseteq C$ be the collection of sequences that have their first nonzero occurrence at an even index. More precisely, define

$$D_n = \{(x_1, x_2, \dots) \in C : x_1 = x_2 = \cdots = x_{n-1} = 0, x_n = 1\}$$

and

$$U = \bigcup_{n\in\mathbb{N}} D_{2n}.$$

Show that U is open.

5. Observe that the sequence $(0, 0, \dots)$ with only zero entries is not in U. Show that if $(x_1, x_2, \dots) \in \overline{U}$, then either $(x_1, x_2, \dots) \in U$ or $(x_1, x_2, \dots) = (0, 0, \dots)$. (Hint: for all k the open neighborhood $U_{x_1,\dots,x_k}$ of $(x_1, x_2, \dots)$ intersects U)

6. Show that $\overline{U} = U \cup \{(0, 0, \dots)\}$.

7. Show that $(0, 0, \dots)$ has no open neighborhood contained in $\overline{U}$ and conclude that $\overline{U}$ is not open.

Bonus exercises

Exercise 5. Let R be a commutative ring and let B be the set of idempotents in R. Recall that we defined a sum $\oplus$ on B given by

$$p_1 \oplus p_2 = p_1 + p_2 - p_1 p_2,$$

where on the right side we mean addition and multiplication in the ring R. Show that the product on B (given by restricting the product on R) distributes over $\oplus$.

Exercise 6. Let A be a C^*-algebra and let $a \in A$. If a is normal, we can define $|a|$ using functional calculus. More generally for $a \in A$ not necessarily normal, we can define $|a| := \sqrt{a^* a}$. Show that this indeed generalizes the above definition in case a is normal. (Hint: $\overline{z}z = |z|^2$)

Exercise 7. Let R be a commutative ring. Show that $\mathrm{Zar}(R)$ is compact.

Exercise 8. Let B be a Boolean algebra. In this exercise we will compare the commutative C^*-algebra $C(\mathrm{Spec}(B))$ corresponding to B with a more algebraic construction of a $*$-algebra $\mathbb{C}[B]$ built from B. Define $\mathbb{C}[B]$ to be the span of formal symbols e_b for $b \in B$ nonzero and product given by $e_b \cdot e_{b'} := e_{bb'}$.

1. Show that $\mathbb{C}[B]$ is an algebra with unit e_1.

2. Show that we can uniquely extend $e_b^* = e_b$ to make $\mathbb{C}[B]$ into a commutative $*$-algebra.

3. Let $I \subseteq B$ be a maximal ideal. Show that

$$e_b \mapsto \begin{cases} 0 & b \in I \\ 1 & b \notin I \end{cases}$$

uniquely extends to an algebra homomorphism $\phi_I \colon \mathbb{C}[B] \to \mathbb{C}$.

4. Given $b \in B$, let $U_b = \{I \in \operatorname{Spec} B : b \notin I\}$ denote the corresponding basic open in the Zariski topology. Show that $e_b \mapsto \mathbb{1}_{U_b}$ extends to a $*$-homomorphism $\mu\colon \mathbb{C}[B] \to C(\operatorname{Spec}(B))$. (Hint: Recall that U_b is clopen, which is needed to show $\mathbb{1}_{U_b}$ is continuous)

5. Show that if $U_b = U_{b'}$ then $b = b'$. Derive that μ is injective.

6. Show that the $*$-subalgebra $\mathbb{C}[B] \subseteq C(\operatorname{Spec}(B))$ separates points. Conclude using Stone-Weierstrass that this $*$-subalgebra is dense.