

# 1 Extra Exercises

*Exercise 1.* Let

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Show that the commutant of  $a$  in  $M_2(\mathbb{C})$  consists of all matrices of the form

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_1 \end{pmatrix}$$

Even though the commutant of any subset  $S$  of an algebra  $A$  is always an algebra, we therefore see that it need not be a  $*$ -algebra.

*Exercise 2.* Let  $S: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  be the Hilbert hotel operator  $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$ . Recall that  $S$  is a bounded linear operator. Let  $\lambda \in \mathbb{C}$  be an element on the unit circle. Consider the following collection of elements of  $\ell^2(\mathbb{N})$  indexed by  $n$

$$x^{(n)} := \frac{1}{\sqrt{n+1}} \left( 1, \frac{1}{\lambda}, \frac{1}{\lambda^2}, \dots, \frac{1}{\lambda^n}, 0, 0, \dots \right)$$

or in other notation if you prefer

$$x_k^{(n)} := \begin{cases} \frac{1}{\lambda^k \sqrt{n+1}} & k \leq n \\ 0 & k > n \end{cases}$$

for  $k \in \mathbb{N} = \mathbb{Z}_{\geq 0}$  the indexing of the sequence.

1. Show that  $\|x^{(n)}\| = 1$  for all  $n$
2. Show that  $\|(S - \lambda)x^{(n)}\| = \frac{\sqrt{2}}{\sqrt{n+1}}$
3. Show that  $\lambda \in \text{Spec } S$  (Hint: use the previous two parts to show that a potential inverse is not bounded).
4. Let  $\mu \in \mathbb{C}$  be in the open unit disk, i.e.  $|\mu| < 1$ . Define

$$y = (1, \lambda, \lambda^2, \dots), \text{ i.e. } y_k = \lambda^k.$$

Show that  $y \in \ell^2(\mathbb{N})$

5. Show that  $S^*y = \lambda y$ .
6. Show that  $\text{Spec}(S)$  is the closed unit disk.

*Exercise 3.* Let  $(S, \mathcal{M}, \mu)$  be a measurable space with a countably additive measure. In this exercise we determine the spectrum of the operator on  $L^2$  given by multiplication with a fixed function  $f$ .

1. Let  $f, f', g, g': S \rightarrow \mathbb{C}$  be measurable. Show that if  $f$  is equal to  $f'$  almost everywhere and  $g$  is equal to  $g'$  almost everywhere, then  $fg$  is equal to  $f'g'$  almost everywhere.

2. Let  $[f] \in L^\infty(S, \mathcal{M}, \mu)$  be the equivalence class of an essentially bounded function  $f: S \rightarrow \mathbb{C}$  (recall that the equivalence relation on  $L^\infty$  is almost everywhere equality). Define a map  $M_f: L^2(S, \mathcal{M}, \mu) \rightarrow L^2(S, \mathcal{M}, \mu)$  on the Hilbert space  $L^2(S, \mathcal{M}, \mu)$  by multiplication by  $f$ . Use the previous exercise to show that this is a well-defined linear map, and moreover that  $M_f$  is a bounded operator of norm  $\|f\|_{L^\infty}$ . (Hint: integrals of functions that are almost everywhere are equal)
3. Suppose  $U \subseteq \mathbb{C}$  is an open neighbourhood of zero such that  $f^{-1}(U)$  has measure zero. Show that  $M_f$  is invertible. (Hint: first try  $f^{-1}(U) = \emptyset$ )
4. Show the contrapositive of the previous question; if  $\mu(f^{-1}(U)) > 0$  for all open neighborhoods  $U$  of 0, then  $M_f$  is not invertible. (Hint: try to show a potential inverse of  $M_f$  would be unbounded. For example, one can work with the indicator function at  $f^{-1}(U)$  for  $U$  the disk of radius  $1/n$  around 0.)
5. Define the essential image of  $f$  to be those  $\lambda \in \mathbb{C}$  such that there exists a neighborhood  $U$  of  $\lambda$  with  $\mu(f^{-1}(U)) > 0$ . Use the previous two parts to show that the spectrum of  $M_f$  is the essential image of  $f$ .

*Exercise 4.* Let  $B$  be a Boolean algebra and  $\mu: B \rightarrow [0, 1]$  a measure

1. Show that  $\mu(b^\perp) = 1 - \mu(b)$  for all  $b \in B$
2. From now on, assume that  $\mu$  maps finite suprema and infima in  $B$  to finite suprema and infima in  $[0, 1]$  with the usual ordering. Show that  $\min\{\mu(b), 1 - \mu(b)\} = 0$  for all  $b \in B$ . Conclude that  $\mu$  takes values in  $\{0, 1\}$ .
3. We use the usual ring structure on the field with two elements  $\{0, 1\} = \mathbb{Z}/2 = \mathbb{F}_2$ . Show that  $\mu(ab) = \mu(a)\mu(b)$ .
4. Show that  $\mu$  is a ring homomorphism (Hint: the product of  $a + b$  and  $ab$  is zero).

*Exercise 5.* Consider the functions  $f_1: \mathbb{C} \rightarrow \mathbb{C}$  and  $f_2: \mathbb{C} \rightarrow \mathbb{C}$  given by  $f_1(x) = 0$  and  $f_2(x) = x$ . Since these are polynomials, continuous functional calculus for  $a \in A$  in a  $C^*$ -algebra  $A$  works as follows:  $f_1(a) = 0$  and  $f_2(a) = a$ .

Now let  $a \in A$  be a nilpotent element. Since  $\text{Spec}(a) = \{0\}$  is contained in the domain of the functions  $f_1$  and  $f_2$ , we can compute  $f_1(a)$  and  $f_2(a)$ . However,  $f_1(x) = f_2(x)$  for all  $x \in \text{Spec}(a)$ . Therefore  $0 = f_1(a) = f_2(a) = a$ , so there are no  $C^*$ -algebras with nilpotent elements.

What went wrong here? Can you fix the mistake and what did we prove instead?

*Exercise 6.* Let

$$a = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{C}).$$

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the map

$$f(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0. \end{cases}$$

1. Note that  $f$  is not continuous. Why can we still define  $f(a)$  using continuous functional calculus?

2. Compute  $f(a)$ .

*Exercise 7.* Let  $(S, \mathcal{M}, \mu)$  be a measurable space with a countably additive measure. We study the projections in the von Neumann algebra  $L^\infty(S, \mathcal{M}, \mu)$ .

1. If  $[p] \in L^\infty(S, \mathcal{M}, \mu)$  is a projection, show that  $p$  is almost everywhere equal to a measurable function  $f: S \rightarrow \mathbb{C}$  such that  $f^2 = f$  and hence  $[p] = [f]$
2. Show that a map  $f: S \rightarrow \{0, 1\}$  is measurable if and only if  $f = 1_M$  is the indicator function for some  $M \in \mathcal{M}$  measurable. Here we make  $\{0, 1\}$  into a measurable space by declaring all subsets to be measurable.
3. Show that  $[1_M] = 0$  in  $L^\infty(S, \mathcal{M}, \mu)$  if and only if  $\mu(M) = 0$ .
4. Show that the map from the Boolean algebra  $(S, \mathcal{M})$  to the Boolean algebra of projections in  $L^\infty(S, \mathcal{M}, \mu)$  is a ring homomorphism.
5. Show that the Boolean algebra of projections in  $L^\infty(S, \mathcal{M}, \mu)$  is isomorphic to  $(S, \mathcal{M})/N_\mu$ , where  $N_\mu$  is the ideal of measure zero subsets.