

Operator algebra course

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1 Introduction Lecture (September 2)

Example 1.1. Let $n \geq 0$ be an integer. $M_n(\mathbb{C})$ is the collection of n by n matrices with entries in \mathbb{C} . From linear algebra we know matrices can be added, multiplied by scalars and we have matrix multiplication with the identity matrix as the unit. There are several matrix norms but one standard one is

$$\|A\| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} \frac{\|Av\|_{\mathbb{C}^n}}{\|v\|_{\mathbb{C}^n}},$$

where $\|v\|_{\mathbb{C}^n}$ is the standard Euclidean norm $|v_1|^2 + \dots + |v_n|^2$. There is a notion $A^* = (\overline{a_{ji}})_{i,j}$ of a conjugate transpose of a matrix $A = (a_{ij})_{i,j}$.

Example 1.2. Let $\ell^\infty(\mathbb{N})$ be the space of bounded sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{C}$. These can be entrywise added and multiplied and the constant sequence 1 is a unit. They also have a norm

$$\|(x_n)\| = \sup_n |x_n|.$$

Moreover, this makes $\ell^\infty(\mathbb{N})$ into a Banach space, i.e. a complete normed vector space. Sequences also have a conjugate operation $(x_n)_n^* = (\overline{x_n})_n$.

Example 1.3. Let $C([0, 1])$ be the space of continuous complex valued functions on a closed interval. These can be added, multiplied and complex conjugated. Since continuous functions on $[0, 1]$ are bounded, the norm

$$\|f\| = \sup_{x \in [0, 1]} |f(x)|$$

is finite.

Definition 1.4. An *algebra* is a complex vector space A equipped with a bilinear multiplication map $\cdot : A \times A \rightarrow A$ which is associative and has a unit $1 \in A$.

Example 1.5. The set $\mathbb{C}[x]$ of polynomials in a single variable x is an algebra. Even though polynomial algebras have a rich geometric theory (algebraic geometry), this is not the type of algebras we consider in this course. Instead we will consider algebras which are more analytic in nature. They will be ‘larger’; they are complete normed spaces.

Definition 1.6. A *normed vector space* is a complex vector space X equipped with a *norm* $\|\cdot\| : X \rightarrow \mathbb{R}_{\geq 0}$ satisfying

$$\|x + y\| \leq \|x\| + \|y\| \quad \|\lambda x\| = |\lambda| \|x\|$$

and $\|x\| = 0$ if and only if $x = 0$.

Every normed vector space X defines a metric space with $d(x, y) = \|x - y\|$.

Definition 1.7. Recall that a *topology* on a set X is a collection \mathcal{T} of subsets of X called *open sets* which are closed under finite intersections and arbitrary unions such that $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ ¹. A *closed set* is the complement of an open set. A set with a topology on it is called a (*topological*) *space*. We say that the topology \mathcal{T}' is *stronger* than \mathcal{T} if $\mathcal{T} \subseteq \mathcal{T}'$.

Example 1.8. If (M, d) is a metric space, then \mathcal{T} given by unions of open balls defines a topology on M . In other words, open balls form a *basis* for the topology on M .

¹In some abstract sense saying that X is closed under empty unions and intersections would already imply $\emptyset, X \in \mathcal{T}$.

Definition 1.9. A(n open) *neighborhood* of $x \in X$ is an open set containing x .

Exercise 1. A subset $U \subseteq X$ is open if and only if every $x \in U$ has an open neighborhood contained in U .

Definition 1.10. A sequence x_n in a topological space *converges to* x if for every open neighbourhood U of x there exists an N such that $x_n \in U$ for all $n > N$.

Exercise 2. A sequence x_n in a metric space *converges to* x if and only if for all $\epsilon > 0$ there exists an N such that $\forall n > N$ we have $d(x, x_n) < \epsilon$.

Definition 1.11. A sequence in a metric space is *Cauchy* if for all $\epsilon > 0$ there exists an N such that $d(x_n, x_m) < \epsilon$ if $n, m > N$. A metric space is *complete* when every Cauchy sequence converges to some element.

Many topologies in analysis can be studied very well using sequences because they are first countable.

Definition 1.12. We say that X is *first countable* if every point $x \in X$ has a countable neighbourhood basis, i.e. there exists a countable collection U_n of neighborhoods of x such that every open neighborhood of x contains U_n for some n .

Example 1.13. Any metric space is first countable with U_n the balls of radius $1/n$.

Lemma 1.14. A subset A of a first countable space X is closed if and only if every sequence a_n in A which converges in X also converges in A .

Proof. ‘ \implies ’ This direction holds in general. Suppose A is closed and a_n converges to $x \in X$. Let us assume $x \in X \setminus A$ and work towards a contradiction. Because $X \setminus A$ is open, there exists an open neighborhood U of x disjoint from A . But by convergence we have that $a_n \in U$ for sufficiently large n , a contradiction.

‘ \impliedby ’ We prove the contrapositive and so assume $X \setminus A$ is not open. Then there is some $x \in X \setminus A$ such that every open neighbourhood U of x intersects nontrivially with A . Pick a countable neighborhood basis U_n of x . Pick an arbitrary $a_n \in U_n \cap A$ for every n . Because U_n is a neighborhood basis, a_n is a sequence of elements of A converging to $x \in X \setminus A$. \square

Definition 1.15. A *Banach space* is a normed vector space which is complete.

Example 1.16. A closed subspace Y of a Banach space X is a Banach space. Indeed, a Cauchy sequence in Y converges in X and its limit is in Y by Lemma 1.14.

Example 1.17. Let $p \geq 1$. Let ℓ^p be the collection of sequences x_n such that $\sum_n |x_n|^p < \infty$. It can be shown to be a Banach space with norm

$$\|x_n\| = \left(\sum_n |x_n|^p \right)^{1/p}.$$

Definition 1.18. A *Banach algebra* is an algebra that is also a Banach space such that $\|ab\| \leq \|a\| \|b\|$. A *C^* -algebra* is a Banach algebra equipped with a ‘star’ operation $*$: $A \rightarrow A$ such that

$$a^{**} = a \quad (ab)^* = b^* a^* \quad (a + b)^* = a^* + b^* \quad (\lambda a)^* = \bar{\lambda} a^*$$

as well as the *C^* -identity*

$$\|a^* a\| = \|a\|^2.$$

Remark 1.19.

1. The C^* -identity is somewhat mysterious. We will prove all sorts of nice consequences of it next week. For example, the norm is completely determined by the algebraic structure. You will also see how some Banach algebras are ‘less nice’ in the exercises.
2. There are obvious notions of algebra over \mathbb{R} , Banach space over \mathbb{R} and Banach algebra over \mathbb{R} . However, whenever we talk about algebras in this document, they will always be over \mathbb{C} . The correct definition of C^* -algebra over \mathbb{R} is rather subtle and will not be considered in this course.
3. The definition of Banach algebra and C^* -algebra also make sense without a unit. In fact, most references don’t assume unitality because there are many examples that don’t satisfy it. However, we will not study non-unital algebras because they will give you a headache.

Example 1.20. The algebra \mathbb{C} with $\|z\| := |z|$ and complex conjugation is a C^* -algebra.

Example 1.21. Let $\{A_i\}_{i \in I}$ be a family of C^* -algebras. Define

$$\bigoplus_i A_i = \{(a_i)_{i \in I} : \sup_{i \in I} \|a_i\| < \infty\}.$$

This is an algebra with pointwise sum, product and scalar multiplication. It has a pointwise $*$ and obvious norm which makes it into a C^* -algebra. In particular, $\ell^\infty(S) := \bigoplus_{s \in S} \mathbb{C}$ is a C^* -algebra for a set S .

Exercise 3. Show that $A = M_2(\mathbb{C})$ with $\|a_{ij}\| := \sum_{i,j} |a_{i,j}|$ is a Banach algebra.

Remark 1.22. Recall that in an algebra A , the unit is unique in the sense that if $1' \in A$ also satisfies $1'a = a = a1'$, then $1 = 11' = 1'$. It follows that in a C^* -algebra, we have that $1^* = 1$ since

$$1^* \cdot a = 1^* \cdot a^{**} = (a^* \cdot 1)^* = a^{**} = a.$$

Definition 1.23. A map $f: X \rightarrow Y$ between topological spaces is *continuous* if for all $U \subseteq Y$ open, $f^{-1}(U)$ is open.

Exercise 4. A map between metric spaces is continuous if and only if it satisfies the usual epsilon delta definition.

Definition 1.24. A space X is *compact* if for all open covers of X — i.e. a family of opens U_i for $i \in I$ such that $\cup_{i \in I} U_i = X$ — there exists a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$.

Lemma 1.25. *Let X be a compact topological space. Then every continuous function $f: X \rightarrow \mathbb{C}$ is bounded.*

Proof. Let $f: X \rightarrow \mathbb{C}$ be continuous. For every $r \in \mathbb{R}$, define the open sets $U_r = \{x \in X : |f(x)| < r\} = f^{-1}(B_r)$. Here B_r is the open disk of radius r in the complex plane. Note that $\bigcup_r U_r = X$ is an open cover of X . Because X is compact, there is a finite subcover $\{U_{r_1}, \dots, U_{r_n}\}$. We see that $|f(x)| \leq \max\{r_1, \dots, r_n\}$ for all $x \in X$. \square

Definition 1.26. If A is a C^* -algebra, we say that a subspace $B \subseteq A$ is a C^* -subalgebra if it is closed in the norm topology, as well as closed under multiplication, the $*$ operation and $1 \in B$.

Since a closed subspace of a Banach space is a Banach space, a C^* -subalgebra is again a C^* -algebra.

Proposition 1.27. *Let X be a compact topological space. Then $C(X)$ is a C^* -algebra.*

Proof. We have seen that $\ell^\infty(X)$ is a C^* -algebra. In Lemma 1.25, we have seen $C(X)$ is a subset of $\ell^\infty(X)$. It therefore suffices to show that it is a closed unital $*$ -subalgebra. Since complex conjugation, addition and multiplication preserves continuity, and the constant function 1 is continuous, this is a $*$ -subalgebra.

It remains to show that $C(X) \subseteq \ell^\infty(X)$ is closed. Because norm spaces are first countable, it suffices to show that for every sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions with limit $f \in \ell^\infty(X)$, the limit is also continuous. This follows by the uniform limit theorem. The short proof on wikipedia is copied here: we have to show that for every $\epsilon > 0$, there exists a neighborhood U of any point $x \in X$ such that:

$$|f(x) - f(y)| < \epsilon, \quad \forall y \in U$$

Consider an arbitrary $\epsilon > 0$. Since the sequence of functions f_n converges uniformly to f by hypothesis, there exists a natural number N such that:

$$|f_N(t) - f(t)| < \frac{\epsilon}{3}, \quad \forall t \in X$$

Moreover, since f_N is continuous, for every x there exists a neighborhood U such that:

$$|f_N(x) - f_N(y)| < \frac{\epsilon}{3}, \quad \forall y \in U$$

In the final step, we apply the triangle inequality in the following way:

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \tag{1}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall y \in U. \tag{2}$$

□

Remark 1.28. Note that $A = \{0\}$ has a unique structure making it into a C^* -algebra with $1 = 0$. This C^* -algebra is additionally the $C(\emptyset)$ and is important for category-theoretic reasons. In many results about C^* -algebras, the C^* -algebra $\{0\}$ is an exception, even when I forget to say so.

Next lecture, we will show continuous operators on a Hilbert space form a C^* -algebra. After that we will need to develop some general theory with the goal of proving an equivalence of categories between compact Hausdorff spaces and commutative C^* -algebras in a few weeks. After that we will move on to von Neumann algebras.

2 Operators on Hilbert space (September 4)

Today we will discuss the main example of a C^* -algebra: continuous operators on a Hilbert space.

We start defining operators on Banach spaces, which already form a Banach algebra. A linear map between infinite-dimensional Banach spaces is not always continuous, but it turns out to be continuous exactly when the following straightforward condition holds.

Definition 2.1. A linear map between Banach spaces $T: X \rightarrow Y$ (also called an *operator*) is *bounded* if there exists a $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$.

Clearly only the zero map is bounded in the literal sense, so the terminology should not be too confusing.

Definition 2.2. Let $B(X, Y)$ denote the bounded operators $X \rightarrow Y$ and $B(X) := B(X, X)$. The *operator norm* on $B(X, Y)$ is defined by

$$\|T\| := \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

The quotient is defined because $\|x\| \neq 0$ if $x \neq 0$. The norm is finite as $\|T\| \leq C$, where C is the bound in the definition of a bounded linear map. Moreover, if $x \neq 0$, then

$$\frac{\|Tx\|}{\|x\|} \leq \sup_{y \in X \setminus \{0\}} \frac{\|Ty\|}{\|y\|} = \|T\| \implies \|Tx\| \leq \|T\|\|x\|$$

and the equation also clearly holds when $x = 0$.

If $T, S \in B(X)$ and $\lambda \in \mathbb{C}$, then $\|\lambda T\| = |\lambda|\|T\|$ and by the triangle equality in X

$$\|T+S\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx + Sx\|}{\|x\|} \leq \sup_{x \in X \setminus \{0\}} \frac{\|Tx\| + \|Sx\|}{\|x\|} \leq \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} + \sup_{y \in X \setminus \{0\}} \frac{\|Ty\|}{\|y\|} = \|T\| + \|S\|.$$

We see that $B(X)$ is a normed space.

Note that $\|\text{id}_X\| = 1$ is bounded. Also note that

$$\|TS\| = \sup_{\|x\|=1} \|TSx\| \leq \sup_{\|x\|=1} (\|T\|\|Sx\|) = \|T\|\|S\|.$$

We conclude that $B(X)$ is a Banach algebra as long as we can prove:

Lemma 2.3. $B(X, Y)$ is complete.

Proof. Let T_n be a Cauchy sequence. The first claim is that for all $x \in X$, the sequence $T_n x$ converges. Indeed, let $x \in X$ and note that the result is trivial unless $x \neq 0$. Now take N so that $\|T_m - T_n\| < \epsilon/\|x\|$. Then

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\| < \epsilon$$

and so $T_m x$ is Cauchy. By completeness of Y , $T_m x$ converges to some element of Y we will call Tx .

To finish, we need to show that T is a bounded linear operator and T_n converges to T . If $x_1, x_2 \in X$, then

$$T(x_1) + T(x_2) = \lim(T_n(x_1) + T_n(x_2)) = T(x_1 + x_2)$$

by linearity of all T_n and the fact that sums of convergent sequences converge to the sum of the limit (continuity of addition). This holds similarly for scalar multiplication, and so T is linear.

Let $\epsilon > 0$. It suffices to show that $\|T - T_n\| < \epsilon$ for n sufficiently large. Indeed, then T is automatically bounded since $T - T_n$ is bounded. Pick N so that $\|T_m - T_n\| < \epsilon/2$ for all $n, m > N$. We claim that $\|T - T_n\| < \epsilon$ if $n > N$. Let $x \in X$. By definition of Tx , we can pick M so that $\|T(x) - T_n(x)\| < \epsilon\|x\|/2$ for all $n > M$. Then if $m > \max\{N, M\}$, we have that

$$\|(T - T_n)(x)\| \leq \|Tx - T_mx\| + \|T_mx - T_nx\| \leq \|Tx - T_mx\| + \|T_m - T_n\|\|x\| < \epsilon\|x\|.$$

We see that $\|T - T_n\| < \epsilon$ for all $n > M$. □

Sometimes the formula

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

is convenient, which we prove as follows. By restricting the supremum to a subset, we have

$$\sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} \geq \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|T(x)\|.$$

Conversely note that when $x \neq 0$

$$\|Tx\| = \left\| \|x\| T\left(\frac{x}{\|x\|}\right) \right\| = \|x\| \|T(x/\|x\|)\|$$

Since $\|x/\|x\|\| = 1$, we thus get

$$\frac{\|Tx\|}{\|x\|} = \|T(x/\|x\|)\| \leq \sup_{\|y\|=1} \frac{\|T(y)\|}{\|y\|},$$

finishing the argument.

Definition 2.4. An *inner product space* is a vector space \mathcal{H} equipped with an inner product. Here an *inner product* is a sesquilinear map

$$\langle -, - \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$$

such that

1. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in \mathcal{H}$ (hence $\langle v, v \rangle \in \mathbb{R}$)
2. $\langle v, v \rangle \geq 0$ for all $v \in \mathcal{H}$
3. $\langle v, v \rangle = 0$ if and only if $v = 0$.

By *sesquilinear*, we mean that

$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \overline{\lambda_1} \langle v_1, w \rangle + \overline{\lambda_2} \langle v_2, w \rangle \quad \langle v, \mu_1 w_1 + \mu_2 w_2 \rangle = \mu_1 \langle v, w_1 \rangle + \mu_2 \langle v, w_2 \rangle.$$

Warning 2.5. Many authors assume instead that inner products are complex linear in the first argument and complex anti-linear in the second, instead of the other way around.

We want to be able to talk about complete inner product spaces. By complete, we will mean that the induced normed space $\|v\| := \sqrt{\langle v, v \rangle}$ is a Banach space. Most of the axioms of a normed space are easy to check, but the triangle equality is not immediately clear. It can be shown as a consequence of the following important lemma (exercise).

Lemma 2.6 (Cauchy–Schwarz). *For all $x, y \in \mathcal{H}$ we have that*

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Proof. You can find one on Wikipedia. □

Definition 2.7. A *Hilbert space* is a (complex) vector space \mathcal{H} with a complete inner product.

Example 2.8. \mathbb{C} is a Hilbert space with $\langle z_1, z_2 \rangle = \bar{z}_1 z_2$.

Definition 2.9. A (*bounded*) (*linear*) *functional* on a Banach space X is a bounded linear map $X \rightarrow \mathbb{C}$. Given a Banach space X , let $X^\vee := B(X, \mathbb{C})$ be the Banach space of bounded linear functionals.²

Example 2.10. Let \mathcal{H} be a Hilbert space and $x \in \mathcal{H}$. The functional $\langle x, - \rangle: \mathcal{H} \rightarrow \mathbb{C}$ is bounded of operator norm $\|x\|$. In other words

$$\|x\| = \sup_{y \in \mathcal{H}} \frac{|\langle x, y \rangle|}{\|y\|}.$$

Since this is obvious for $x = 0$, we assume $x \neq 0$. Indeed, by Cauchy-Schwarz we have that

$$\|x\| \leq \frac{|\langle x, y \rangle|}{\|y\|}$$

for all nonzero $y \in \mathcal{H}$. Moreover, the supremum is attained by $y = x$ as

$$\|x\| = \frac{\langle x, x \rangle}{\|x\|}.$$

The following lemma requires some work to prove.

Lemma 2.11. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. If $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded operator, there exists a unique operator $T^*: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that*

$$\langle Tv, w \rangle = \langle v, T^*w \rangle$$

for all $v \in \mathcal{H}_1$ and $w \in \mathcal{H}_2$.

Assuming the lemma, however, it is not that difficult to show:

Theorem 2.12. *Let \mathcal{H} be a Hilbert space. Then $B(\mathcal{H})$ is a C^* -algebra.*

²It is more common to denote the dual space by X^* , but in the context of Hilbert spaces this could cause confusion with the $*$ operation on bounded operators on Hilbert spaces we will soon discuss. An alternative would have been the physics convention (which I like): denote the star operation in a C^* -algebra by \dagger .

Proof. We have already shown $B(\mathcal{H})$ is a Banach algebra. By using the uniqueness property of the adjoint, we see that $(T + S)^* = T^* + S^*$ and $(\lambda T)^* = \bar{\lambda}T^*$ by sesquilinearity of the inner product. Also note that $T^{**} = T$ since

$$\langle T^*v, w \rangle = \overline{\langle w, T^*v \rangle} = \overline{\langle Tw, v \rangle} = \langle v, Tw \rangle.$$

Finally $(ST)^* = T^*S^*$ is also clear.

So it remains to show the C^* -identity. For this, first note that by Example 2.10, we have for any operator T that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, \|y\|=1} |\langle Tx, y \rangle|.$$

Applying this to T^*T , we see that

$$\|T^*T\| = \sup_{\|x\|=1, \|y\|=1} |\langle T^*Tx, y \rangle| = \sup_{\|x\|=1, \|y\|=1} |\langle Tx, Ty \rangle|.$$

Now we have on the one hand that

$$\sup_{\|x\|=1, \|y\|=1} |\langle Tx, Ty \rangle| \geq \sup_{\|x\|=1} |\langle Tx, Tx \rangle| = \sup_{\|x\|=1} \|Tx\|^2 = \left(\sup_{\|x\|=1} \|Tx\| \right)^2 = \|T\|^2$$

while on the other hand

$$\sup_{\|x\|=1, \|y\|=1} |\langle Tx, Ty \rangle| \leq \sup_{\|x\|=1, \|y\|=1} \|Tx\| \|Ty\| = \|T\|^2.$$

□

For the proof of the existence of adjoints, we need a classical theorem of Hilbert space theory.

Theorem 2.13 (Riesz-Fréchet). *The map $RF: \mathcal{H} \rightarrow \mathcal{H}^\vee$ defined by sending x to $\langle x, - \rangle$ is a complex antilinear isomorphism of Banach spaces.*

Here a map $T: X \rightarrow Y$ is called *complex antilinear* if $T(x_1 + x_2) = Tx_1 + Tx_2$ and $T(\lambda x) = \bar{\lambda}Tx$. It still makes sense to talk about boundedness for such maps.

The fact that RF is complex antilinear follows from sesquilinearity of the inner product. It follows from Exercise 2.10 that the RF is an *isometry*, i.e. $\|Tx\| = \|x\|$. An isometry is in particular bounded. Injectivity then follows because every isometry is injective. So the surjectivity, i.e. the fact that every bounded functional $f: \mathcal{H} \rightarrow \mathbb{C}$ is of the form $\langle x, - \rangle$ for some $x \in \mathcal{H}$ is the nontrivial part. This fact is proven in³ [6, Number 5] using the theory of projections.

Now, given the Riesz-Fréchet theorem, we can define T^*x as the inverse of the functional

$$y \mapsto \langle x, Ty \rangle,$$

under the Riesz-Fréchet isomorphism. In other words, uniqueness follows by injectivity of RF and existence by surjectivity. We can conclude that $B(\mathcal{H})$ is a C^* -algebra.

If I is a (potentially uncountable) set and $(x_i)_{i \in I}$ a collection of elements in a Banach space X , we say that $\sum_{i \in I} x_i$ *converges to the value* $x \in X$ if for every $\epsilon > 0$ there exists a finite set $F_0 \subseteq I$ such that $\|\sum_{i \in F} x_i - x\| < \epsilon$ whenever F contains F_0 .

³The section numbering in Bram's PhD thesis is potentially confusing. There are sections, subsections and subsubsections as usual. But deeper than that there are also 'numbers' which are then divided into roman numerals which I will refer to as 'paragraphs'. So Number 3 Paragraph V refers to the definition of direct sum of C^* -algebras.

Example 2.14. Let \mathcal{H}_i for $i \in I$ be a family of Hilbert spaces. Then the *direct sum Hilbert space* is

$$\bigoplus_{i \in I} \mathcal{H}_i = \{v_i \in \mathcal{H}_i : \sum_{i \in I} \|v_i\|_{\mathcal{H}_i}^2 < \infty\}$$

with inner product

$$\langle (v_i)_{i \in I}, (w_i)_{i \in I} \rangle := \sum_{i \in I} \langle v_i, w_i \rangle.$$

This is shown in [6, Number 6 Paragraph II]. As a subexample, we can take all \mathcal{H}_i to be \mathbb{C} . We obtain that the Banach space $\ell^2(S)$ is a Hilbert space with inner product

$$\langle (x_n), (y_n) \rangle = \sum_n \overline{x_n} y_n.$$

Remark 2.15. We have now introduced direct sums of Hilbert spaces and of C^* -algebras. These two concepts are distinct. Firstly note that for infinite direct sums, the finiteness condition on Hilbert spaces is stronger. For example

$$\ell^2(\mathbb{N}) = \bigoplus_{\mathbb{N}}^{\text{Hilb}} \mathbb{C} \subsetneq \bigoplus_{\mathbb{N}}^{\text{C}^* \text{ Alg}} \mathbb{C} = \ell^\infty(\mathbb{N}).$$

Moreover, $\ell^2(\mathbb{N})$ is not an algebra as it is not even preserved under product.

3 The category of C^* -algebras (September 9)

Definition 3.1. Let A be a C^* -algebra. An element $a \in A$ is called

1. *invertible* if there exists a (necessarily unique) $a^{-1} \in A$ such that $a^{-1}a = 1 = aa^{-1}$.
2. *self-adjoint* if $a^* = a$
3. *unitary* if a^* is the inverse of a

Example 3.2. For $A = M_n(\mathbb{C})$, this reproduces the notions of self adjoint and unitary matrix you already know.

Example 3.3. In $C(X)$, we have that $f \in C(X)$ is self adjoint if and only if it is real valued. A function is unitary if and only if it is valued in the unit circle.

Definition 3.4. Let A, B be C^* -algebras. A **-homomorphism* $\phi: A \rightarrow B$ is a (unital) algebra homomorphism such that $\phi(a^*) = \phi(a)^*$.

Remark 3.5. It might be surprising that we didn't require ϕ to be bounded. The reason is that this will be automatic, but we don't have the machinery to prove this yet.

Example 3.6. If A is a nonzero C^* -algebra, there is a unique **-homomorphism* $\mathbb{C} \rightarrow A$ that sends $1 \mapsto 1_A$.

We do a small detour reviewing the basics of category theory. This will be useful later to express Gelfand duality. Indeed, Gelfand duality will say how commutative C^* -algebras are 'the same' as compact Hausdorff⁴ topological spaces, but how to we express this?

Definition 3.7. A *category* \mathcal{C} consists of a class $\text{ob } \mathcal{C}$ called *objects*, for all $x, y \in \text{ob } \mathcal{C}$ a set $\text{Hom}_{\mathcal{C}}(x, y)$ of *morphisms* and for all $x, y, z \in \text{ob } \mathcal{C}$ a composition operation

$$\text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x, z) \quad (g, f) \mapsto g \circ f$$

which is associative and has units $\text{id}_x \in \text{Hom}_{\mathcal{C}}(x, x)$.

When we will write $x \in \mathcal{C}$, we actually mean $x \in \text{ob } \mathcal{C}$. Sometimes we will write $f: x \rightarrow y$ for $f \in \text{Hom}_{\mathcal{C}}(x, y)$. We will also often omit the subscript \mathcal{C} from the notation if it's clear what category we are talking about.

In a category, we often want to identify two objects not only when they are equal, but also when they are isomorphic:

Definition 3.8. We say that a morphism $f \in \text{Hom}_{\mathcal{C}}(x, y)$ is an *isomorphism* if it has a both-sided inverse $f^{-1} \in \text{Hom}_{\mathcal{C}}(y, x)$ under composition. Two objects are called *isomorphic* if there exists an isomorphism between them.

Example 3.9. The category of groups and group homomorphisms. Isomorphisms are group isomorphisms.

Example 3.10. The category pt is defined to have a single object and a single morphism.

Example 3.11. The category of Banach spaces and bounded operators. Isomorphisms are operators with a bounded inverse.

Another option would be to take morphisms to be norm-preserving. Then isomorphisms would be isometric isomorphisms. We will focus on the category of bounded operators in this document.

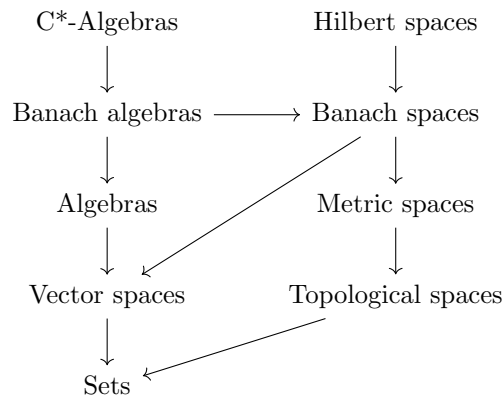
⁴We will see what Hausdorff spaces are later.

Example 3.12. The category of C^* -algebras and $*$ -homomorphisms. Recall that in the category of algebras (or rings or groups), a homomorphism is an isomorphism if and only if it is a bijection. This similarly applies to $*$ -homomorphisms; the inverse of a bijective $*$ -homomorphism will again be a $*$ -homomorphism.

Example 3.13. The category of topological spaces and continuous maps. Isomorphisms are called *homeomorphisms*.

Example 3.14. If \mathcal{C} is a category, the *opposite category* \mathcal{C}^{op} has the same objects, but the composition is reversed. In other words, $\text{Hom}_{\mathcal{C}^{\text{op}}}(x, y) = \text{Hom}_{\mathcal{C}}(y, x)$ with composition in the reversed order.

Example 3.15. There are many ‘forgetful functors’ that are rather boring and simply forget some structure. For example, there is a functor from the category of groups to the category of sets that takes the underlying set. This is fine since every group homomorphism has an underlying map of sets. This also gives a way to compare the many concepts we have already encountered in this course by a bunch of forgetful functors:



Also note that this diagram commutes. We have also learned about a way to go from Hilbert spaces to C^* -algebras by taking bounded operators. However, if we would draw that arrow in the top square would not commute; $B(\mathcal{H})$ is different from \mathcal{H} as a Banach space unless $\mathcal{H} = \mathbb{C}$.

So how do we express the relationship between the category of topological spaces and the category of C^* -algebras? First we need to know what is a ‘morphism between categories’.

Definition 3.16. If \mathcal{C}, \mathcal{D} are categories, a *functor* $\mathcal{C} \rightarrow \mathcal{D}$ consists of an assignment $F: \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$ on objects, and a family of maps $F_{x,y}: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$ for all objects $x, y \in \text{ob } \mathcal{C}$ such that $F_{x,z}(g \circ f) = F_{y,z}(g) \circ F_{x,y}(f)$ and $F_{x,x}(\text{id}_x) = \text{id}_{Fx}$.

We usually abuse notation and write F for $F_{x,y}$.

Example 3.17. The assignment $X \mapsto C(X)$ defines a functor from the category of compact topological spaces to the opposite of the category of C^* -algebras. Indeed, you can verify that if $g: X \rightarrow Y$ is a continuous map, then $f \mapsto f \circ g$ defines a $*$ -homomorphism $C(Y) \rightarrow C(X)$. Moreover, this respects identities and composition.

Example 3.18. If \mathcal{C} is a category and $x \in \mathcal{C}$ is an object, there is a functor $\text{pt} \rightarrow \mathcal{C}$ sending the unique object to x .

How do we say that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ expresses that \mathcal{C} are ‘the same’? We could say that F has an inverse functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G = \text{id}_{\mathcal{D}}$ and $G \circ F = \text{id}_{\mathcal{C}}$. This notion is called

isomorphism of categories, but we usually want something slightly weaker. For this, we will see that functors $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ are themselves objects of a category. In that case, it might be better to talk about isomorphisms of functors as opposed to equality. Morphisms in the category of functors are called *natural transformations*.

Definition 3.19. If $F_1, F_2: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ are functors, a natural transformation $F_1 \Rightarrow F_2$ is an assignment of $\phi_x: F_1(x) \rightarrow F_2(x)$ to $x \in \text{ob } \mathcal{C}_1$ such that the diagram

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\phi_x} & F_2(x) \\ \downarrow F_1(f) & & \downarrow F_2(f) \\ F_1(y) & \xrightarrow{\phi_y} & F_2(y) \end{array}$$

commutes for all $f \in \text{Hom}_{\mathcal{C}_1}(x, y)$. Here a diagram *commutes* if it doesn't matter in which direction one traverses it. So the above diagram means $F_2(f) \circ \phi_x = \phi_y \circ F_1(f)$.

One can show that natural transformations assemble into a category with composition $(\psi \circ \phi)_x = \psi_x \circ \phi_x$. Now we can say that F_1 and F_2 are naturally isomorphic if they are isomorphic in this category. This is the same as saying that there exist a natural transformation $F_1 \Rightarrow F_2$ of which the components $\phi_x: F_1(x) \rightarrow F_2(x)$ are isomorphisms for all $x \in \mathcal{C}_1$.

Definition 3.20. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence* if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to $\text{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $\text{id}_{\mathcal{D}}$.

A functor is an equivalence of categories if and only if it is 'fully faithful' and 'essentially surjective'. These conditions are good to know about as they are generally easier to check than to directly verify the existence of the inverse G , but we probably won't need them in this course.

Definition 3.21. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a functor. An object $x \in \text{ob } \mathcal{C}$ equipped with a collection of morphisms $f_i: x \rightarrow D(i)$ for all $i \in \text{ob } \mathcal{I}$ is called a *limit* of D if it has *universal property* of being a *cone*: for all morphisms $\phi: i_1 \rightarrow i_2$ in \mathcal{I} , we have that

$$\begin{array}{ccc} & x & \\ f_{i_1} \swarrow & & \searrow f_{i_2} \\ D(i_1) & \xrightarrow{D(\phi)} & D(i_2) \end{array}$$

commutes⁵ and for any other object $y \in \text{ob } \mathcal{C}$ equipped with a collection of morphisms $g_i: y \rightarrow D(i)$ for all $i \in \text{ob } \mathcal{I}$ such that the analogous diagram commutes, there exists a unique morphism $g: x \rightarrow y$ such that $f_i \circ g = g_i$ for all $i \in \text{ob } \mathcal{I}$.

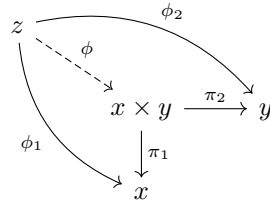
There is a completely analogous notion of *colimit* in which all the arrows are reversed.

Let I be a set and let $\mathcal{I} = \sqcup_{i \in I} \text{pt}$ be the category with object set I , with only identity morphisms. Let \mathcal{C} be a category. Then a functor $F: \mathcal{I} \rightarrow \mathcal{C}$ exactly picks out an I -family of objects $x_i \in \text{ob } \mathcal{C}$ for $i \in I$. A *product* of the x_i , denoted $\prod_{i \in I} x_i$, is a limit of F .

For example, if $|I| = 2$, a product of x and y is an object $x \times y$ together with morphisms $\pi_1: x \times y \rightarrow x$ and $\pi_2: x \times y \rightarrow y$ such that for every pair of morphisms $\phi_1: z \rightarrow x$ and $\phi_2: z \rightarrow y$,

⁵Look at the diagram, it's a cone!

there exists a unique morphism $\phi: z \rightarrow x \times y$ such that $\pi_1\phi = \phi_1$ and $\pi_2\phi = \phi_2$. This information is usually summarized by the commutative diagram:



Example 3.22. A product in the category of sets is given by the usual Cartesian product. A product in the category of groups is the usual product of groups.

Example 3.23. A product of topological spaces X_i in the category of topological spaces and continuous maps, is the product topology on $\prod_{i \in I} X_i$. This topology is generated by sets the form $\prod_{i \in I} U_i$, where $U_i \subseteq X$ are open and $U_i = X$ for all but finitely many $i \in I$.

Let \mathcal{I} be the category consisting of two objects a and b and two morphisms $a \rightarrow b$ next to the two identity morphisms $a \rightarrow a$ and $b \rightarrow b$. Then a functor $F: \mathcal{I} \rightarrow \mathcal{C}$ is the same data as two objects $x, y \in \mathcal{C}$ and two morphisms $f, g: x \rightarrow y$. An *equalizer* of f and g is the limit of F .

Example 3.24. A equalizer of $f, g: G \rightarrow H$ in the category of groups is given by the kernel of fg^{-1} .

4 Series in Banach spaces and holomorphic functions (September 11)

In this lecture we will develop some analytic tools. This might be somewhat dry if you don't like analysis as much (like me), but it will be useful for understanding the spectrum of an element $a \in A$, and then the spectrum of a C^* -algebra A .

4.1 Convergent series

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements in a Banach space X . We denote $\sum_{n=0}^{\infty} x_n := \lim_{k \rightarrow \infty} \sum_{n=0}^k x_n$ if the series converges.

Definition 4.1. We say that the series $\sum_{n \in \mathbb{N}} x_n$ converges absolutely if

$$\sum_n \|x_n\| < \infty.$$

Lemma 4.2. If a series in a Banach space converges absolutely, then it converges.

Proof. If x_n converges absolutely, then $(\sum_{n=0}^N x_n)_N$ is a Cauchy sequence because for $N_1 < N_2$ we have

$$\left\| \sum_{n=0}^{N_2} x_n - \sum_{n=0}^{N_1} x_n \right\| = \left\| \sum_{n=N_1}^{N_2} x_n \right\| \leq \sum_{n=N_1}^{N_2} \|x_n\|$$

and $(\sum_{n=0}^N \|x_n\|)_N$ is a Cauchy sequence since it converges. By completeness $(\sum_{n=0}^N x_n)_N$ converges. \square

Let A be a Banach algebra. The fact that small elements have a geometric series will be used often.

Lemma 4.3. Let $a \in A$ satisfy $\|a\| < 1$. Then

1. $\sum_{n=0}^{\infty} a^n$ converges absolutely
2. $\sum_{n=0}^{\infty} a^n$ is the inverse of $1 - a$.

Proof. By the formula for geometric series in the real numbers, we get

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} < \infty.$$

Since $(\sum_{n=0}^{\infty} a^n)_N$ converges absolutely it converges to some element by Lemma 4.2. To verify it converges to $(1 - a)^{-1}$, we take the limit of the equation

$$(1 - a)(1 + a + a^2 + \cdots + a^N) = 1 - a^{N+1},$$

as $N \rightarrow \infty$, which by continuity of left multiplication by $1 - a$ gives us

$$(1 - a) \left(\sum_n a^n \right) = 1,$$

since $\|a^N\| \leq \|a\|^N \rightarrow 0$ as $N \rightarrow \infty$. We can derive

$$\left(\sum_n a^n \right) (1 - a) = 1$$

in a similar manner. □

Remark 4.4. For people who haven't seen geometric series: let $0 < r < 1$ be a real number. Note that

$$(1 - r)(1 + r + r^2 + \dots + r^N) = 1 - r^{N+1},$$

and so

$$\sum_{n=0}^N r^n = \frac{1 - r^{N+1}}{1 - r}$$

for every N . Thus, since r^N converges to 0 (because $r < 1$), we see that

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

We need more generalities about convergence of series in Banach spaces. The following definitions and results are probably familiar in the special case that the Banach space is \mathbb{C} . For the next few lemmas, let x_n be a sequence in a Banach space, thought of as a series.

Definition 4.5. The *radius of convergence* of (x_n) is $R(x_n) := (\limsup_n \|x_n\|^{1/n})^{-1} \in [0, \infty]$.

Lemma 4.6. Suppose that $\sum_{n=0}^{\infty} x_n$ converges. Then $\lim x_n = 0$.

Proof. By convergence $\left(\sum_{n=0}^k x_n \right)_k$ is a Cauchy sequence. Let $\epsilon > 0$ and pick N large enough so that

$$\left\| \sum_{n=0}^k x_n - \sum_{n=0}^l x_n \right\| < \epsilon$$

if $k, l > N$. Now take $l = k + 1$ to see that $\|x_n\| < \epsilon$ if $n > N$. □

Proposition 4.7 ([6, 13II]). Let $z \in \mathbb{C}$.

1. If $|z| < R(x_n)$, then $\sum_n x_n z^n$ converges absolutely.

2. If $\sum_n x_n z^n$ converges, then $|z| \leq R(x_n)$.

Proof. Suppose that $|z| < R(x_n)$. We must show that

$$\sum_m \|x_m\| |z|^m < \infty.$$

If $z = 0$, this is obvious, so assume that $|z| > 0$. Then since $R(x_n) > 0$, we have that $R(x_n)^{-1} |z| < 1$. We can therefore pick $\epsilon > 0$ small enough so that $(R(x_n)^{-1} + \epsilon) |z| < 1$. Now since $\limsup_m \|x_m\|^{\frac{1}{m}} <$

$R(x_n)^{-1} + \varepsilon$, we can take N so large that $\|x_m\|^{\frac{1}{m}} \leq R(x_n)^{-1} + \varepsilon$ for all $m \geq N$. Then $\|x_m\|^{\frac{1}{m}} |z| \leq (R(x_n)^{-1} + \varepsilon) |z| < 1$ for all $m \geq N$, and so

$$\begin{aligned} \sum_m \|x_m\| |z|^m &= \sum_{m=0}^{N-1} \|x_m\| |z|^m + \sum_{m=N}^{\infty} \left(\|x_m\|^{\frac{1}{m}} |z| \right)^m \\ &\leq \sum_{m=0}^{N-1} \|x_m\| |z|^m + \sum_{m=N}^{\infty} \left((R(x_n)^{-1} + \varepsilon) |z| \right)^m < \infty \end{aligned}$$

by convergence of the geometric series.

Suppose now instead that $\sum_n x_n z^n$ converges. Then $\|x_n\| |z|^n$ converges to 0 by Lemma 4.6. In particular, there is N with $\|x_m\| |z|^m \leq 1$ for all $m \geq N$. Then $\|x_m\|^{\frac{1}{m}} \leq |z|^{-1}$ for all $m \geq N$, so that $R(x_n)^{-1} = \limsup_m \|x_m\|^{\frac{1}{m}} \leq |z|^{-1}$, giving $|z| \leq R(x_n)$. \square

4.2 Integration

We will define integrals in Banach spaces. There will be no surprises, but it is also somewhat tedious. For this we need a lemma which is also useful in many other contexts. To define the terms in the lemma, we make a brief topological detour:

Definition 4.8. A subset Y of a topological space X is called *dense* if $\overline{Y} = X$. Here we used the notation \overline{A} for the *closure* of a subset $A \subseteq X$ of a topological space X ; the intersection of all closed sets in X which contain A .

Exercise 5. Show that an arbitrary intersection of closed sets in a topological space X is closed. Conclude that the closure of a subset $A \subseteq X$ is closed.

Lemma 4.9. Let $D \subseteq X$ be a dense subspace of a normed space X and let Y be a Banach space. If $T: D \rightarrow Y$ is a bounded linear map, then T extends uniquely to an operator $T': X \rightarrow Y$ with the same operator norm.

Proof. Let $x \in X$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in D converging to x . If T' exists, we must have

$$T'x = \lim_{x_n \rightarrow x} Tx_n.$$

Since x_n converges in X , x_n is a Cauchy sequence in D . Because T is bounded, it follows that Tx_n is a Cauchy sequence and so this limit exists.

Note first that T' is well-defined. Indeed, if x'_n is another sequence converging to x , then $x_n - x'_n$ converges to 0, so that $\|Tx_n - Tx'_n\|$ converges to 0. From this it follows that

$$\lim_{x_n \rightarrow x} Tx_n = \lim_{x'_n \rightarrow x} Tx'_n$$

as desired.

We show that T' is bounded with the same norm as⁶ as follows

$$\|T'x\| = \left\| \lim_{n \rightarrow \infty} T(x_n) \right\| = \lim_{n \rightarrow \infty} \|T(x_n)\| \leq \lim_{n \rightarrow \infty} \|T\| \|x_n\| = \|T\| \|x\|$$

\square

⁶If x_n converges to x , then $\|x\| = \lim_{n \rightarrow \infty} \|x_n\|$ by the reverse triangle inequality. In other words, $\|\cdot\|: X \rightarrow \mathbb{R}$ is a continuous map.

Let X be a Banach space and take a half open⁷ interval $(s, t]$ in \mathbb{R} . Define a *step function* $f: (s, t] \rightarrow X$ to be a linear combination of *indicator functions* of subintervals. Here a subinterval is a subset $(s_1, t_1] \subseteq (s, t]$ for some $s \leq s_1 < t_1 \leq t$ and the indicator function on a set $S \subseteq (s, t]$ with value $x \in X$ is the function

$$f(t') = \begin{cases} x & \text{if } t' \in S, \\ 0 & \text{if } t' \notin S. \end{cases}$$

If f is a step function, it is possible to pick $s = s_0 < s_1 < \dots < s_k < s_{k+1} = t$ for some $k \geq 0$ so that $f(t')$ takes a fixed value $x_i \in X$ whenever $s_i < t' \leq s_{i+1}$ for all $i \in \{0, \dots, k\}$. Moreover, given f there is a unique minimal choice of s_j (in the sense that k is smallest) where we have that $x_i \neq x_{i+1}$ for all i . The integral of this step function is then defined as

$$\int_s^t f = \sum_{i=0}^k x_i (s_{i+1} - s_i) \in X.$$

Every step function has a canonical extension to $[s, t]$ and from now on we will consider them as functions on $[s, t]$. Step functions form a linear subspace $S[s, t] \subseteq \ell^\infty([s, t]; X)$, where $\ell^\infty([s, t]; X)$ is the Banach space of bounded maps $[s, t] \rightarrow X$ with the supremum norm $\sup_{t' \in [s, t]} \|f(t')\|$.

The integral defines a linear map $\int_s^t: S[s, t] \rightarrow X$ which is bounded because

$$\left\| \int_s^t f(t') dt' \right\| \leq (t - s) \|f\|.$$

It therefore extends to a bounded linear map $\int_s^t: \overline{S[s, t]} \rightarrow X$ by Lemma 4.9. Let $C([s, t]; X) \subseteq \ell^\infty([s, t]; X)$ the subspace of continuous functions $[s, t] \rightarrow X$. Step functions are rarely continuous, but it turns out that we do have $C([s, t]; X) \subseteq \overline{S}$:

Exercise 6. Show that every continuous function $[s, t] \rightarrow X$ is a uniform limit of step functions. (Hint: by the Heine–Cantor theorem, such a function is uniformly continuous)

Let $\Gamma \subseteq \mathbb{C}$ be a *contour*, i.e. a subset that can be parametrized by a C^1 path $\gamma: [s, t] \rightarrow \mathbb{C}$. The *contour integral* of a continuous map $f: \mathbb{C} \rightarrow X$ is defined by

$$\int_\Gamma f = \int_s^t f(\gamma(t')) \gamma'(t') dt'$$

One can check that this is independent of the choice of parametrization using the chain rule and the fact that any reparameterization corresponds to a C^1 map $\tau: [s_1, t_1] \rightarrow [s_2, t_2]$.

4.3 Holomorphic functions

Functions which are complex differentiable are usually called holomorphic functions. Holomorphic functions from the complex numbers into a Banach space make sense. Even more, we will see later that all cool complex analysis theorems such as Cauchy's integral theorem still hold.

Definition 4.10. Let X be a Banach space. If $U \subseteq \mathbb{C}$ is an open set, a function $f: U \rightarrow X$ is *holomorphic* if for all $z \in U$

$$f'(z) := \lim_{w \rightarrow z} \frac{f(z) - f(w)}{z - w}$$

exists.

⁷We do this for technical reasons. It is also possible to allow intervals of all four types.

Exercise 7.

1. Given $x \in X$, show that the map $\mathbb{C} \rightarrow X$ given by $z \mapsto zx$ is holomorphic with derivative the constant function at x .
2. Show that if $U, V \subseteq \mathbb{C}$ are open, $f: V \rightarrow X$ is holomorphic and $g: U \rightarrow V$ is holomorphic in the usual complex analysis sense, then $f \circ g$ is holomorphic.
3. Show that if A is a Banach algebra and $f: U \rightarrow A$ and $g: U \rightarrow A$ are holomorphic, then $f \cdot g: U \rightarrow A$ is holomorphic.

5 The spectrum of an element of a C^* -algebra (September 16)

The spectrum is an infinite dimensional generalization of the collection of eigenvalues of a matrix.

Definition 5.1. Let A be a complex algebra and $a \in \mathbb{C}$. Then the *spectrum of a* is

$$\text{Spec}_A a = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not invertible}\}.$$

The spectrum makes sense in any algebra, but becomes much nicer for Banach algebras. For example, the spectrum is always compact and nonempty. We will also use spectral techniques to show that the norm on a C^* -algebra is unique. The rest of the lecture we will work towards these results.

Remark 5.2. Let $B \subseteq A$ be a subalgebra and let $b \in B$. It is clear that if b is invertible as an element of B , then it also is invertible in A , but the converse is not clear. This implies that we have an inclusion $\text{Spec}_A b \subseteq \text{Spec}_B b$. For general (even Banach) algebras, this inclusion can be strict. But we will see later that this is an equality for C^* -subalgebras of C^* -algebras and hence we will from now on omit the subscript A in $\text{Spec}_A a$.

Example 5.3. Let $a \in M_n(\mathbb{C})$ be a matrix. By the rank nullity theorem, $a - \lambda$ is invertible if and only if $\ker(a - \lambda) = 0$. This happens if and only if λ is an eigenvalue of a . We see that $\text{Spec } a$ is the collection of eigenvalues of a .

Example 5.4. Let X be a compact Hausdorff space. Note that $f \in C(X)$ is invertible exactly when it does not take the value 0. It follows that the spectrum of $f \in C(X)$ is $f(X)$. One intuition for this, is that we can think of $f \in C(X)$ as a ‘diagonal $X \times X$ matrix’ with $f(x)$ on the diagonal.

Remark 5.5. The term spectrum comes from quantum physics; the spectrum of an observable operator are the possible measurement outcomes.

Next week we will see how the following lemma generalizes to any continuous function on a C^* -algebra.

Proposition 5.6. *Let A be an algebra. Let p be a polynomial and $a \in A$. Then $\text{Spec } p(a) = p(\text{Spec } a)$.*

Proof. Let $\lambda \in \text{Spec } a$ and assume $a - \lambda$ is not right invertible (the case of left invertible is analogous). To prove that $p(\lambda) \in \text{Spec } p(a)$, it suffices to show that $p(a) - p(\lambda) = (a - \lambda)b$ for some $b \in B$. Indeed, if $p(a) - p(\lambda)$ would be invertible, then we get that $b(p(a) - p(\lambda))^{-1}$ is a right inverse of $a - \lambda$.

Now note that if $p(x)$ is the polynomial $\sum_{k=0}^n \mu_k x^k$, we have

$$p(a) - p(\lambda) = \mu_n(a^n - \lambda^n) + \dots + \mu_1(a - \lambda).$$

To finish this direction, we observe that for $k > 0$

$$a^k - \lambda^k = (a^{k-1} + \lambda a^{k-2} + \dots + \lambda^{k-1})(a - \lambda).$$

Conversely, suppose $\gamma \in \text{Spec } p(a)$ and use the fundamental theorem of algebra to write the polynomial $p(x) - \gamma$ as $(x - \zeta_1) \dots (x - \zeta_n)$. Since in the product

$$p(a) - \gamma = (a - \zeta_1) \dots (a - \zeta_n)$$

the order does not matter, we see that $a - \zeta_i$ is not invertible for some i . We obtain that $\zeta_i \in \text{Spec } a$ and so $p(\zeta_i) = \gamma \in p(\text{Spec } a)$. \square

Lemma 5.7. *Let A be a Banach algebra and $A^\times := \{a \in A : a \text{ invertible}\}$.*

1. $A^\times \subseteq A$ is open
2. if $a \in A$, then $\text{Spec } A \subseteq \mathbb{C}$ is closed

Proof. (1) Let $a \in A^\times$. We want to show that a has an open neighborhood contained in A^\times . Note that if $b \in A^\times$, then $ab \in A^\times$ since $(ab)^\times = b^\times a^\times$. We therefore have that left multiplication by a is a continuous map $L_a: A^\times \rightarrow A^\times$ with continuous inverse $L_{a^{-1}}: A^\times \rightarrow A^\times$. We see that L_a is a homeomorphism. As $1 \in A$ is clearly invertible, L_a maps open neighborhoods of 1 to open neighborhoods of a ⁸. Now we observe that Lemma 4.3 precisely says that the open ball around $1 \in A$ of radius 1 is contained in A^\times . We see that the image of this open ball under L_a is an open neighborhood of a .

(2) We show that $\mathbb{C} \setminus \text{Spec } a$ is open. Suppose $a - \lambda$ is invertible for some fixed $\lambda \in \mathbb{C}$. We need to show that as $|\lambda - \lambda_0|$ becomes small for $\lambda_0 \in \mathbb{C}$, then $a - \lambda_0$ is eventually invertible. However, because A^\times is open, we know that $a - b$ becomes invertible as $\|b\|$ becomes small for $b \in A$. So we finish after applying this to the case where $b \in \mathbb{C} \subseteq A$ after observing that then $\|b\| = |b|$. \square

Proposition 5.8. *Let A be a Banach algebra and $a \in A$. Then*

1. $\text{Spec } a$ is bounded by the closed disk with radius $\|a\|$;
2. $\text{Spec } a$ is compact.

Proof. We showed that $\text{Spec } a$ is closed in Lemma 5.7. Since every closed and bounded subset of \mathbb{C} is compact by the Heine-Borel theorem, part 2 follows from part 1.

We therefore only have to show that $a - \lambda$ is invertible if $|\lambda| > \|a\|$. Since $\lambda \neq 0$, we can apply Lemma 4.3 to get that $1 - a/\lambda$ is invertible. This clearly implies $a - \lambda$ is invertible. \square

The most important result about the spectrum of an element of a Banach algebra A is the *spectral radius formula*. The proof requires quite some techniques involving the existence and uniqueness of power series associated to A -valued holomorphic functions. We postpone the proofs of these techniques to the end of this section.

Theorem 5.9 (Beurling's theorem). *Let A be a Banach algebra and let $a \in A$ be arbitrary. Then the sequence $\|a^n\|^{1/n}$ converges. Its limit is*

$$\sup\{|\lambda| : \lambda \in \text{Spec } a\}.$$

Proof. To prove that a sequence x_n in \mathbb{R} converges to x , it suffices to show that $\limsup_n x_n \leq x \leq \liminf_n x_n$.

Let $\lambda \in \text{Spec } a$. By Proposition 5.6, we have $\lambda^n \in \text{Spec } a^n$ and so

$$|\lambda|^n = |\lambda^n| \leq \|a^n\| \implies |\lambda| \leq \|a^n\|^{1/n}.$$

We obtain

$$\sup\{|\lambda| : \lambda \in \text{Spec } a\} \leq \liminf_n \|a^n\|^{1/n}.$$

So we are done if we can prove

$$\sup\{|\lambda| : \lambda \in \text{Spec } a\} \geq \limsup_n \|a^n\|^{1/n}.$$

⁸General continuous maps do not map open sets to open sets, but homeomorphisms do.

Define $f(z) := (1 - az)^{-1}$, which makes sense for those $z \in \mathbb{C}$ such that $1 - az$ is invertible. Since for $z = 0$ it clearly is, let us assume $z \neq 0$. Then $(1 - az)^{-1}$ is invertible if and only if $z^{-1}(1 - az) = z^{-1} - a$ is invertible, i.e. if $z^{-1} \notin \text{Spec } a$. The resulting set

$$U := \{z \in \mathbb{C} : 1/z \notin \text{Spec } a\}$$

on which f is defined, is open because $z \mapsto 1/z$ is continuous on $\mathbb{C} \setminus \{0\}$ and $\text{Spec } a$ is closed. Note that $f(z)$ is holomorphic on U , since $\lambda \mapsto (a - \lambda)^{-1}$ is holomorphic outside of $\text{Spec } a$ by Lemma 5.15. Indeed, recall from Exercise 7 that $z \mapsto z$ is holomorphic, products of holomorphic functions are holomorphic, and compositions of holomorphic functions on subsets of \mathbb{C} with holomorphic A -valued functions are holomorphic. We can apply this to $\lambda(a - \lambda)^{-1} = (a/\lambda - 1)^{-1}$ and use the holomorphic function $\lambda \mapsto 1/\lambda = z$ on $\mathbb{C} \setminus \{0\}$.

It follows by Lemma 4.3 that the series

$$\sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az)^n \quad (3)$$

converges absolutely to $f(z)$ if $\|az\| = \|a\|\|z\| < 1$. Since the desired inequality is obvious in case $\|a\| = 0$, we may assume $\|a\| > 0$ without loss of generality.

It follows by Proposition 5.18 that the series expansion of $(1 - az)^{-1}$ holds more generally for all $z \in U$ contained in the disk with radius the radius of convergence of 3, i.e. such that $|z| < R = \limsup_{n \in \mathbb{N}} \|a^n\|^{1/n}$. Moreover, it follows by part 2 of Proposition 4.7 that the power series $\sum_{n=0}^{\infty} a^n z^n$ diverges if $|z| > \limsup_{n \in \mathbb{N}} \|a^n\|^{1/n}$, and so U can't contain such a larger disk. Therefore, if $r < \limsup_n \|a^n\|^{1/n}$, there exists a $\lambda \in \text{Spec } a$ such that $|\lambda| > r$. \square

Lemma 5.10. *Let A be a C^* -algebra. Suppose $a \in A$ is self adjoint and $\sum_{n=0}^{\infty} a^n$ converges. Then $\|a\| < 1$.*

Proof. By the C^* -identity, we inductively obtain

$$\|a^{2^n}\| = \|(a^{2^{n-1}})^* a^{2^{n-1}}\| = \|(a^{2^{n-1}})\|^2 = \dots = \|a\|^{2^n}.$$

Since $(\sum_{n=0}^k a^n)_{k \in \mathbb{N}}$ is a Cauchy sequence, the limit $\lim_{n \rightarrow \infty} \|a^n\|$ exists and is zero. But then the subsequence $(\|a\|^{2^n})_{n \in \mathbb{N}}$ also converges to 0 and hence $\|a\| < 1$. \square

Corollary 5.11. *Let A be a C^* -algebra and let $a \in A$ be self adjoint. Then*

$$\|a\| = \sup\{|\lambda| : \lambda \in \text{Spec } a\}.$$

More generally

$$\|a\| = \sqrt{\sup\{|\lambda| : \lambda \in \text{Spec } a^*a\}}$$

for arbitrary $a \in A$.

Proof. In the final step of the proof of the spectral radius formula, we now see that $\sum_n a^n z^n$ diverges even if $|z| > \|a\|$ by the previous lemma. We therefore get $\sup_{\lambda \in \text{Spec } a} |\lambda| < \|a\|$

For the final formula, simply use that a^*a is self adjoint to get

$$\|a\|^2 = \|a^*a\| = \sup\{|\lambda| : \lambda \in \text{Spec } a^*a\}.$$

\square

Remark 5.12. Let A be a C^* -algebra. If a self adjoint $a \in A$ has $\text{Spec } a = \{0\}$, then $a = 0$. On the other hand, any nilpotent element $a \in A$ has $\text{Spec } a = \{0\}$ too. So nontrivial nilpotent elements of a C^* -algebra are never self adjoint.

Theorem 5.13. *Let A, B be C^* -algebras and let $\phi: A \rightarrow B$ be a $*$ -homomorphism with $B \neq 0$. Then ϕ is bounded with $\|\phi\| = 1$.*

Proof. Let $a \in A$. Observe that $\text{Spec}(\phi(a')) \subseteq \text{Spec } a'$ for all $a' \in A$. Therefore we have that

$$\|\phi(a)\|^2 = \|\phi(a)^* \phi(a)\| = \|\phi(a^* a)\| = \sup_{\lambda \in \text{Spec } \phi(a^* a)} |\lambda| \leq \sup_{\lambda \in \text{Spec } a^* a} |\lambda| = \|a^* a\| = \|a\|^2,$$

so that $\|\phi(a)\| \leq \|a\|$ for all $a \in A$. On the other hand, we have that $\|\phi(1)\| = \|1\| = 1$ since $B \neq 0$ and so

$$\sup_{a \in A \setminus \{0\}} \frac{\|\phi(a)\|}{\|a\|} = 1.$$

□

Missing proofs

Let A be a Banach algebra. The first thing we will show is that $\lambda \mapsto (a - \lambda)^{-1}$ is a holomorphic function $\mathbb{C} \setminus \text{Spec}(a) \rightarrow A$.

Lemma 5.14. *Let A be a Banach algebra. The map $\text{inv}: A^\times \rightarrow A^\times$ given by $a \mapsto a^{-1}$ is continuous.*

Proof. Note that inv is not a linear operator and so we can't simply show that this map is bounded.

We first prove that inv is continuous around $1 \in A^\times$. In other words, we have to show that $(1 - a)^{-1}$ gets closer to $\text{inv}(1) = 1$ as $1 - a$ gets closer to 1, i.e. as $\|a\| \rightarrow 0$. Assume $\|a\| < 1$ so that we can use our favorite formula for $(1 - a)^{-1}$. We obtain by geometric series

$$\|(1 - a)^{-1} - 1\| = \left\| \sum_{n=1}^{\infty} a^n \right\| \leq \sum_{n=1}^{\infty} \|a\|^n = \frac{\|a\|}{1 - \|a\|},$$

which approaches zero as $\|a\| \rightarrow 0$ as desired.

To prove continuity at an arbitrary $b \in A^\times$, note first that $\text{inv} = R_{b^{-1}} \circ \text{inv} \circ L_{b^{-1}}$, where $R_a: A^\times \rightarrow A^\times$ is right multiplication by $a \in A$. Continuity of inv at b now follows from continuity of $R_{b^{-1}}$, continuity of $L_{b^{-1}}$, and continuity of inv at $1 \in A$. □

Proposition 5.15. *Let A be a Banach algebra and $a \in A$. The function $R_a: \lambda \mapsto (a - \lambda)^{-1}$ is holomorphic on the complement of the closed subset $\text{Spec } a$ of \mathbb{C} .*

Proof. Let us show R_a is holomorphic at $\lambda \in \text{Spec } a$. We expect that $-(a - \lambda)^{-2}$ is the derivative of R_a . Pick $\mu \in \text{Spec } a$ distinct from λ . We have that

$$\frac{(a - \mu)^{-1} - (a - \lambda)^{-1}}{\mu - \lambda} = \frac{(\lambda - \mu)(a - \mu)^{-1}(a - \lambda)^{-1}}{\mu - \lambda} = -(a - \mu)^{-1}(a - \lambda)^{-1}. \quad (4)$$

Here we used that (slightly sketchy notation to make it more believable)

$$\frac{1}{a - \mu} - \frac{1}{a - \lambda} = \frac{\mu - \lambda}{(a - \mu)(a - \lambda)},$$

which follows by subtracting the two equations

$$\begin{aligned}(a - \lambda)^{-1} &= (a - \lambda)^{-1}(a - \mu)^{-1}(a - \mu) \\ (a - \mu)^{-1} &= (a - \mu)^{-1}(a - \lambda)^{-1}(a - \lambda)\end{aligned}$$

from each other. It follows that the expression 4 converges to $-(a - \lambda)^{-1}$ as $\mu \rightarrow \lambda$ by continuity of multiplication by $(a - \lambda)^{-1}$ and of $x \mapsto x^{-1}$. \square

Next we needed some results about holomorphic functions and their power series.

Proposition 5.16. *Let (x_n) be a sequence in a Banach space X . Define $U = \{z \in \mathbb{C} : z < R(x_n)\}$ and*

$$f(z) := \sum_n z^n x_n.$$

Then f is holomorphic with derivative at $z \in U$ the absolutely convergent series

$$\sum_{n=1}^{\infty} n z^{n-1} x_n = \sum_{n=0}^{\infty} (n+1) z^n x_{n+1}$$

with the same radius of convergence.

Proof. Note that for $z \in U$, the element $f(z) \in X$ is well-defined as the series converges absolutely by Lemma 4.7. Rewriting

$$(n+1)^{1/n} \|x_{n+1}\|^{1/n} = (n+1)^{1/n} \|x_{n+1}\|^{\frac{n+1}{n}} \|x_{n+1}\|^{1/n},$$

note that the first two terms converge to 1 and the last one has lim sup equal to R^{-1} . We see using $\|(n+1)z^n x_{n+1}\| = (|z|(n+1)^{1/n} \|x_{n+1}\|^{1/n})^n$ as before that $\sum_{n=0}^{\infty} (n+1)z^n x_{n+1}$ converges absolutely with the same radius of convergence.

A tedious analysis argument now shows this series is a derivative of f at z . The proof is given for C^* -algebras in [6, Proposition 13 IV], but immediately generalizes. \square

Corollary 5.17. *Let x_n be a sequence in a Banach space X . Suppose (x_n) has positive radius of convergence and $\sum_n x_n z^n = 0$ for all $|z| < r \leq R(x_n)$. Then $x_n = 0$ for all n .*

Proof. Let $f(z) := \sum_n x_n z^n$ for $\text{dom } f = \{|z| < r\}$. Then for all $z \in \text{dom } f$ we have $|z| < R(x_n)$ so that $f(z)$ converges absolutely to 0. Then we have $f(0) = x_0 = 0$

Clearly we have $f'(z) = 0$ for f the zero function. On the other hand by the previous proposition $f'(z) = \sum_n n z^{n-1} x_n$ absolutely. It follows that $f'(0) = x_1 = 0$. Hence we are done by induction. \square

Proposition 5.18. *Let $f: U \rightarrow A$ be a holomorphic function and suppose that we are given a sequence $a_n \in A, w \in \mathbb{C}$ and $r > 0$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - w)^n$$

for all z in the open disk of radius r around w . If the series converges in a disk of larger radius $R > r$ around w still contained in U , then the series still converges to $f(z)$ for z in that disk.

Proof. The proof uses a generalization of the Cauchy integral formula for Banach spaces. One proof is given in [6, Proposition 15VII] as a consequence of the Cauchy integral formula [6, Proposition 15VI] and Corollary 5.17. \square

6 Positive elements of a C^* -algebra (September 18)

The main topic of this lecture is to define when a self adjoint element of a C^* -algebra A is called positive. This will use that define a partial order on the set of self adjoint elements.

Definition 6.1. A self adjoint element $a \in A$ is called *positive* if $\text{Spec}(a) \subseteq [0, \infty)$.

A slightly annoying feat of this definition is that 0 is positive. (Indeed, $0^* = 0$ and $0 - \lambda$ is invertible exactly when $\lambda \neq 0$.) Maybe the above should have been called ‘non-negative’, but we are stuck with the above terminology. In particular, consider the following example.

Example 6.2. If $A = M_n(\mathbb{C})$, an element $a \in A$ is positive exactly when it is ‘positive semidefinite’. A positive definite matrix is instead a self-adjoint matrix with spectrum contained in $(0, \infty)$.

Example 6.3. The matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has spectrum contained in $[0, \infty)$. However, it is not positive because it is not self-adjoint.

Remark 6.4. The product of positive elements need not be positive. Again, a counterexample can be constructed in $M_2(\mathbb{C})$ already.

Not any self adjoint element is positive, but its spectrum is always real:

Lemma 6.5 (10 XIII and XV). *If $a \in A$ is self adjoint, then $\text{Spec } a \subseteq \mathbb{R}$.*

Proof. Suppose $z = \mu + i\lambda \in \mathbb{C} \setminus \mathbb{R}$. Our goal is to show that $a - z$ is invertible. Since $a - \mu$ is self adjoint, we can assume $\mu = 0$ without loss of generality.

To show that $a - i\lambda$ is invertible, we want to pick N large and write

$$a - i = (a + iN) - (iN + i\lambda).$$

Then we can use Proposition 5.8 to conclude that $a - i\lambda$ is invertible if $\|a + iN\| < \|iN + i\lambda\| = N + \lambda$. Here we assumed that $\lambda > 0$; otherwise replace N by $-N$ in the rest of the proof. To show that N can indeed be chosen like that, note that

$$\|a + iN\|^2 = \|(a + iN)^*(a + iN)\| = \|a^2 + N^2\| \leq \|a\|^2 + N^2.$$

Therefore it suffices to pick N so large that

$$\|a\|^2 + N^2 < (N + \lambda)^2 = N^2 + 2N\lambda + \lambda^2.$$

This is possible because of the growth of the $2N\lambda$ term on the right hand side since $\lambda > 0$. □

The above fact also allows us to prove spectral permanence, which I promised last week:

Proposition 6.6 (spectral permanence). *Let $B \subseteq A$ be a sub C^* -algebra and let $b \in B$. Then $\text{Spec}_B b = \text{Spec}_A b$.*

Proof. Since the other inequality is obvious, we show that $\text{Spec}_B b \subseteq \text{Spec}_A b$. So we need to show that if $b - \lambda$ has an inverse in A for some $\lambda \in \mathbb{C}$, then this inverse lies in B . It suffices to take $\lambda = 0$.

We will now see why it suffices to prove this for b self adjoint. Indeed, suppose that we have proven the result for self adjoints and let $b \in B$ be an arbitrary element with inverse $c \in A$. Then

$c^* \in A$ is the inverse of $b^* \in B$ and so $cc^* \in A$ is the inverse of $b^*b \in B$. Since b^*b is self adjoint, we see that $cc^* \in B$, but then we also have that $c = c \cdot (c^*b^*) = (cc^*) \cdot b^*$ lies in B .

So it remains to prove that if $b \in B$ is self adjoint and invertible in A , then it's invertible in B . For this, note that since $\text{Spec } b \subseteq \mathbb{R}$ by Lemma 6.5, the sequence $b + i/n$ consists of only invertible elements of B . As the sequence converges to b and inv is continuous, the sequence $(b + i/n)^{-1}$ converges to $b^{-1} \in A$. Because B is a closed subset of A , we see that $b^{-1} \in B$. \square

Definition 6.7. Define a relation \leq on self-adjoint elements by

$$a \leq b \iff b - a \text{ is positive.}$$

The following lemma will be useful to prove that \leq is a partial order.

Lemma 6.8 (17 V). *Let $a \in A$ be self adjoint. Then the following are equivalent*

1. a is positive.
2. For all $t \geq \|a\|/2$ we have $\text{Spec}(a) \subseteq [0, 2t]$.
3. There exists $t \geq \|a\|/2$ such that $\text{Spec}(a) \subseteq [0, 2t]$.
4. There exists $t \geq \|a\|/2$ such that $\|a - t\| \leq t$.

Proof. (2) \implies (3) and (3) \implies (1) are trivial. We now prove (1) \implies (2). Suppose $\text{Spec}(a) \subseteq [0, \infty)$ and let $t \geq \|a\|/2$. Then if $\lambda \in \text{Spec}(a)$, we have that $|\lambda| \leq \|a\| \leq 2t$.

It remains to show that (3) \iff (4). It suffices to prove that for all $t \geq 0$ we have $\text{Spec}(a) \subseteq [0, 2t]$ iff $\|a - t\| \leq t$. Note first that $\text{Spec}(a - t) \subseteq \mathbb{R}$ by Lemma 6.5 since $a - t$ is self adjoint. It follows by Corollary 5.11 and the fact that $\text{Spec}(a - t) = \text{Spec}(a) - t$ that

$$\|a - t\| = \sup\{|\lambda - t| : \lambda \in \text{Spec } a\}$$

We obtain that $\|a - t\| \leq t$ iff $|\lambda - t| \leq t$ for all $\lambda \in \text{Spec}(a)$ iff $\lambda \in [0, 2t]$ for all $\lambda \in \text{Spec}(a)$. \square

Proposition 6.9 (9 VII). *The relation \leq is a partial order.*

Proof. Reflexivity follows from the fact that $0 \in A$ is positive.

For transitivity, note that it suffices to show that $a_1 + a_2$ is positive if a_1 and a_2 are positive. Indeed, suppose $b_1 \leq b_2$ and $b_2 \leq b_3$. Then $b_2 - b_1$ and $b_3 - b_2$ are positive and so it follows that

$$b_3 - b_1 = (b_3 - b_2) + (b_2 - b_1)$$

is positive.

So assume $a_1, a_2 \geq 0$. Using (4) of the lemma above, pick $t_1 \in \mathbb{R}$ with $\|a_1 - t_1\| \leq t_1$ and $t_1 \geq \|a_1\|/2$ as well as $t_2 \in \mathbb{R}$ with $\|a_2 - t_2\| \leq t_2$ and $t_2 \geq \|a_2\|/2$. Then

$$\|a_1 + a_2 - (t_1 + t_2)\| \leq \|a_1 - t_1\| + \|a_2 - t_2\| \leq t_1 + t_2$$

and $\|a_1 + a_2\|/2 \leq \|a_1\|/2 + \|a_2\|/2 \leq t_1 + t_2$.

For antisymmetry, note first that if $a \geq 0$ and $a \leq 0$, then $\text{Spec } a = \{0\}$ and so $a = 0$. Therefore if $a \geq b$ and $b \geq a$, then $a - b \geq 0$ and $a - b \leq 0$ and hence $a = b$ \square

Lemma 6.10. *Let a be a self-adjoint element of A and $\lambda \in [0, \infty)$. We have $-\lambda \leq a \leq \lambda$ iff $\|a\| \leq \lambda$ for $\lambda \in [0, \infty)$.*

Proof. By the spectral radius formula and the fact that a has real spectrum, we have that $\|a\| \leq \lambda$ if and only if $\text{Spec } a \subseteq [-\lambda, \lambda]$. On the other hand, we have that $a \leq \lambda$ exactly when $\text{Spec } a \subseteq (-\infty, \lambda]$. \square

It follows that $\|a\| = \inf\{\lambda \in \mathbb{R}_{\geq 0} : -\lambda \leq a \leq \lambda\}$. So the norm on self adjoint elements is also completely determined by the order.

Lemma 6.11. *Let A be a nonzero C^* -algebra. If $u \in A$ is unitary, then $\text{Spec } u$ is contained in the unit circle.*

Proof. By the C^* -identity and the fact that $A \neq 0$, we have that $\|u\| = 1$. So $\text{Spec } u$ is contained in the unit disk. If $a \in A$ we have that $\text{Spec } a^* = \{\bar{\lambda} : \lambda \in \text{Spec } a\}$. If a is invertible, then $\text{Spec } a^{-1} = \{\lambda^{-1} : \lambda \in \text{Spec } a\}$. Since $u^{*-1} = u$, we see that if $\lambda \in \text{Spec } u$, then $|\bar{\lambda}^{-1}| \leq 1$ and hence $|\lambda| \geq 1$. \square

7 The spectrum of a commutative C^* -algebra (September 23)

In previous lectures we have learned that if X is a compact topological space, then $C(X)$ is a commutative C^* -algebra. Coming lectures, we will learn about an ‘inverse’ to this procedure; the *Gelfand spectrum of a commutative Banach algebra*. This is such a fundamental concept in modern mathematics, that I will make a brief conceptual detour.

The general philosophy in modern geometry is that there is always some correspondence between ‘spaces’ and ‘algebras of functions’. In differential geometry for example, mathematicians study smooth manifolds and real-valued infinitely differentiable functions on them.

In classic algebraic geometry, mathematicians study ‘varieties’, which are essentially spaces of solutions to algebraic equations. Associated to the space \mathbb{C}^n , algebraic geometers assign the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, because they believe that those are the ‘functions on \mathbb{C}^n ’. More generally, if $p \in \mathbb{C}[x_1, \dots, x_n]$ is a polynomial, the zero set

$$X = \{x \in \mathbb{C}^n : p(x) = 0\}$$

has as its corresponding function algebra the quotient

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{(p)},$$

where (p) is the ideal generated by p . For example, you can see that adding to some polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ a multiple of p will not change its values on the zero set X . Therefore it determines the same function on X . We already see the first indication that ideals inside a ring correspond to certain subspaces, and so special ideals might correspond to points. These special ideals will be the *maximal ideals*, essentially because the points of the space are the minimal subspaces (there is some duality/contravariance here).

Remark 7.1 (Lightning summary of ideal types for those who forgot). Let R be a ring. An ideal $I \subseteq R$ is called *maximal* if it is proper and for any other proper ideal $I \subseteq J \subseteq R$, we have that $I = J$. An ideal is called *prime* if for all $x, y \in R$ we have that $xy \in I$ implies that $x \in I$ or $y \in I$. An ideal is maximal if and only if R/I is a field and prime if and only if R/I is an integral domain.

Modern algebraic geometry was developed in large parts by Alexander Grothendieck. His original field was functional analysis, from where he learned about the concept of a spectrum. He realized that the concept applied to an arbitrary ring and this started a whole new point of view on algebraic geometry.

So if X is a nonempty⁹ compact space, how do we find back the space X in the C^* -algebra $C(X)$?¹⁰ The idea is as follows. If $x \in X$, then we can

1. define a map $\text{ev}_x : C(X) \rightarrow \mathbb{C}$, which is easily checked to be a $*$ -homomorphism.
2. define the vanishing ideal

$$\ker \text{ev}_x = \{f \in C(X) : f(x) = 0\}.$$

⁹The empty compact space corresponds to the zero algebra, and will therefore be subject to similar annoying exclusions in theorems that I will mostly ignore.

¹⁰In high generality, the correspondence between algebra and geometry is not as simple as here. It might happen for example that we have an interesting space X on which all global functions vanish. In that case, we would have to start talking about the *sheaf* of functions on X , i.e. the assignment $U \mapsto C(U)$ for opens $U \subseteq X$. We won’t need that in our operator algebraic setting though.

Note that by the first isomorphism theorem, we have that ev_x induces an isomorphism $C(X)/\ker \text{ev}_x \cong \mathbb{C}$. Indeed ev_x is surjective, because the constant functions on X are always continuous. It follows from the fact that \mathbb{C} is a field that $\ker \text{ev}_x$ is a maximal ideal.

We have thus come to the conclusion that given a commutative Banach algebra A , we should try to study

1. algebra homomorphisms $\phi: A \rightarrow \mathbb{C}$ (sometimes called a *character*)
2. maximal ideals $I \subseteq A$.

The latter makes sense for an arbitrary commutative ring and so we define:

Definition 7.2. The *spectrum* of a commutative ring R is the collection of its maximal ideals.

Next we will show that for commutative Banach algebras maximal ideals are in bijection with algebra homomorphisms $\phi: A \rightarrow \mathbb{C}$. For this we will perform a brief study of ideals in (not necessarily commutative) Banach algebras. We will see that closed ideals are useful as quotients by them give again a Banach algebra.

Lemma 7.3. *Let A be a Banach algebra.*

1. *The closure of a proper ideal is again a proper ideal.*
2. *Any maximal ideal $I \subseteq A$ is closed.*

Proof. The closure $\bar{I} \subseteq A$ of an ideal $I \subseteq A$ is again an ideal by continuity of multiplication. Now assume that $I \subseteq A$ is a proper ideal. It suffices to show that the ball of radius 1 around $1 \in A$ is contained in $A \setminus I$. So suppose there would be an $a \in I$ such that $\|1 - a\| < 1$. Then $a^{-1} \in A$ exists and hence $a^{-1}a = 1 \in I$, contradicting the fact that I is proper.

By maximality of I , we see that either I is closed or $\bar{I} = A$. The latter option is excluded by part 1. \square

Exercise 8. Let $a \in [0, 1]$ and define

$$I = \{f \in C([0, 1]) : f(x) = 0 \text{ in some neighborhood of } a\}.$$

Show that I is an ideal of $C([0, 1])$, but it's not closed.

In showing that the quotient of a Banach algebra by a closed ideal is again a Banach algebra, the most tedious step will be showing that the quotient of a Banach space by a closed subspace is again a subspace. We therefore formulate some lemmas to work towards that first. The following converse of Lemma 4.2 will be useful for this.

Lemma 7.4. *Let $(X, \|\cdot\|)$ be a normed linear space. Suppose that every series $\sum_{k=1}^{\infty} x_k$ which converges absolutely also converges in X . Then X is a Banach space.*

Proof. Let $(x_n)_{n \geq 1}$ be a Cauchy sequence in X . We will show it converges.

First note that we can choose a strictly increasing sequence of indices $(n_k)_{k \geq 1}$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k} \quad (k \geq 1).$$

Indeed, pick n_1 arbitrary; since (x_n) is Cauchy, pick $n_2 > n_1$ with $\|x_{n_2} - x_{n_1}\| \leq 2^{-1}$. Having chosen n_k , choose $n_{k+1} > n_k$ such that $\|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}$.

We will now show the subsequence (x_{n_k}) converges. Consider the series

$$\sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}).$$

Because we have

$$\sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$$

the series is absolutely convergent, hence convergent by hypothesis. Note that the partial sums telescope:

$$\sum_{k=1}^m (x_{n_{k+1}} - x_{n_k}) = x_{n_{m+1}} - x_{n_1} \xrightarrow{m \rightarrow \infty} \sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}),$$

so x_{n_m} converges to $L := x_{n_1} + \sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k})$ as $m \rightarrow \infty$.

Since any Cauchy sequence with a convergent subsequence is convergent itself, we have shown that every Cauchy sequence in X converges, so X is complete. \square

Lemma 7.5. *Let X be a Banach space and $Y \subseteq X$ a closed subspace. Then the vector space X/Y has the structure of a Banach space and the projection map $q: X \rightarrow X/Y$ is continuous of norm 1.*

Proof. Define a norm on the quotient by

$$\|[x]\| := \inf_{y \in Y} \|x + y\|.$$

Note how this is independent of the representative $x \in X$ of $[x] \in X/Y$. One can directly verify the triangle inequality and $\|\lambda[x]\| = |\lambda| \|[x]\|$. If $\|[x]\| = 0$, we can take a sequence $y_n \in Y$ such that $\|y_n - x\| < 1/n$. Since $Y \subseteq X$ is closed, we obtain that $x \in Y$ and so $[x] = 0$. So X/Y is a normed space. The fact that $X \rightarrow X/Y$ is bounded of norm 1 follows immediately as

$$\inf_{y \in I} \|x + y\| \leq \|x\|$$

by taking $y = 0$.

For completeness of X/Y , we will apply Lemma 7.4. Let $\sum_{k=1}^{\infty} u_k$ be a series in X/Y with $\sum_{k=1}^{\infty} \|u_k\| < \infty$. For each k , choose a representative $x_k \in X$ of u_k so that

$$\|x_k\| \leq \|u_k\| + 2^{-k}.$$

Then

$$\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} (\|u_k\| + 2^{-k}) = \sum_{k=1}^{\infty} \|u_k\| + 1 < \infty.$$

Hence the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent in X , and since X is Banach, it converges in X to some $x \in X$.

By linearity of q ,

$$q\left(\sum_{k=1}^n x_k\right) = \sum_{k=1}^n q(x_k) = \sum_{k=1}^n u_k \quad (n \in \mathbb{N}),$$

and passing to the limit as $n \rightarrow \infty$ using continuity of q yields

$$\sum_{k=1}^{\infty} u_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n u_k = \lim_{n \rightarrow \infty} q \left(\sum_{k=1}^n x_k \right) = q \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n x_k \right) = q(x) \in X/Y.$$

Thus every absolutely convergent series in X/Y converges. By the lemma, X/Y is complete, hence a Banach space. \square

Proposition 7.6. *Let $I \subseteq A$ be a closed ideal of a Banach algebra. Then the quotient ring A/I has the structure of a Banach algebra.*

Proof. It follows by Lemma 7.5 that A/I is a Banach space. We know that the quotient of an algebra by an ideal is an algebra. So to show that this defines a Banach algebra, we only need to compute

$$\begin{aligned} \|[a]\| \| [b] \| &= \inf_{y_1 \in I} \| a + y_1 \| \inf_{y_2 \in I} \| b + y_2 \| = \inf_{(y_1, y_2) \in I \times I} \| a + y_1 \| \| b + y_2 \| \\ &\geq \inf_{(y_1, y_2) \in I \times I} \| ab + ay_2 + y_1b + y_1y_2 \| \\ &\geq \inf_{y \in I} \| ab + y \| = \| [ab] \|. \end{aligned}$$

The second inequality follows because $ay_2 + y_1b + y_1y_2 \in I$. \square

It follows by Proposition 7.6 and Lemma 7.5 that for any closed ideal $I \subseteq A$, the projection map $A \rightarrow A/I$ is a continuous algebra homomorphism of operator norm 1.

Theorem 7.7 (Gelfand-Mazur). *If A is a Banach algebra which is a division ring, then the unit map $\mathbb{C} \rightarrow A$ is an isomorphism of Banach algebras.*

Proof. If $a \in A$, then we know that $\text{Spec } a \neq \emptyset$ by the spectral radius formula. Hence, there is a $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible. Since A is a division ring, we see that $a - \lambda = 0$. \square

Proposition 7.8. *Let A be a Banach algebra. There is a bijection between the set Φ_A of maximal ideals of A and the algebra homomorphisms $A \rightarrow \mathbb{C}$.*

Proof. If $\phi: A \rightarrow \mathbb{C}$ is an algebra homomorphism, then it induces an isomorphism $A/\ker \phi \cong \mathbb{C}$. Since \mathbb{C} is a field, $\ker \phi$ is a maximal ideal.

Conversely, if $I \subseteq A$ is a maximal ideal, then by Lemma 7.3 it is closed. By Proposition 7.6, A/I is a Banach algebra. Since I is maximal and hence prime, A/I is a division ring and so by the Gelfand-Mazur theorem we get that $A/I \cong \mathbb{C}$. Using the projection to the quotient, we get an algebra homomorphism $A \rightarrow A/I \rightarrow \mathbb{C}$.

It is easy to verify that these operations are inverse to one another. \square

It follows that every algebra homomorphism $A \rightarrow \mathbb{C}$ is bounded of norm 1.

8 The Gelfand transform (September 25)

Given a commutative ring R , the proposal was to assign to R the collection of its maximal ideals. In case $R = A$ is a commutative Banach algebra, we have shown this set is in collection with Φ_R , the collection of all algebra homomorphisms $A \rightarrow \mathbb{C}$. We will call this set the *Gelfand spectrum* and will soon give it a topology. We will now study in case how to map A to $C(\Phi_A)$. This is completely tautological, but also very confusing:

Definition 8.1. Let A be a commutative Banach algebra. The *Gelfand transform* \hat{a} of $a \in A$ is the map $\Phi_A \rightarrow \mathbb{C}$ given by evaluation at a .

Example 8.2. Let $A = C(X)$ for X a compact space, and $ev_x \in \Phi_A$ the evaluation homomorphism $A \rightarrow \mathbb{C}$ corresponding to a fixed $x \in X$. Then \hat{f} for $f \in A$ maps x to $f(x)$. When X is Hausdorff (defined below), we will see later there are no other maximal ideals in $C(X)$, and that this will give us $C(X) \cong C(\Phi_A)$.

Convinced by our definition of \hat{a} , we now ask the question: can we define a topology on Φ_A that makes $\hat{a}: \Phi_A \rightarrow \mathbb{C}$ into a continuous map? The answer to this question is trivially yes, because we can always equip it with the ‘universal’ topology that makes all \hat{a} continuous for $a \in A$. In other words, we can ask $\hat{a}^{-1}(U)$ to be open for all opens $U \subseteq \mathbb{C}$ and all $a \in A$ and generate the weakest possible topology. In topology this general construction is called the *initial topology*, but variations of this construction are often called a *weak topology* in analysis.

Definition 8.3. Let X be a set and $f_i: X \rightarrow Y_i$ a collection of maps to a family of topological spaces Y_i , $i \in I$. Then the *weak topology* generated by the f_i is the topology generated by the sets $f_i^{-1}(U)$ as $U \subseteq Y_i$ goes through the open sets of Y and i goes through I .

Note that for a fixed $i \in I$, the collection of sets of the form $f_i^{-1}(U)$ for $U \subseteq Y_i$ open is closed under finite intersections and arbitrary unions. However, a general open set in a weak topology is not necessarily of the form $f_i^{-1}(U)$ for some $U \subseteq Y_i$ as it might combine various indices $i \in I$ instead.

Example 8.4. The *subspace topology* on a subset X of a topological space Y is the weak topology generated by the inclusion. In other words, it consists exactly of the sets $U \cap X$ for $U \subseteq Y$ open.

Example 8.5. Given a family of topological spaces Y_i , the *product topology* on the set $\prod_{i \in I} Y_i$ is the weak topology with respect to all projection maps $\pi_j: \prod_{i \in I} Y_i \rightarrow Y_j$.

Definition 8.6. A topological space X is *Hausdorff* (or T_2) if any two distinct points can be separated by opens. More specifically, this means that for all $x, y \in X$ distinct there exist open neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Example 8.7. The indiscrete topology on a set of at least two elements is not Hausdorff. The cofinite topology on an infinite set is not Hausdorff. Every metric space is Hausdorff.

Spaces in this course will almost always be Hausdorff.

The following is a basic useful result about Hausdorff spaces.

Proposition 8.8. *Let X be a compact Hausdorff space. If $Y \subseteq X$ is closed, then it is compact.*

We will not prove the following theorem.

Theorem 8.9 (Tychonoff). *If $(X_i)_{i \in I}$ is a family of compact spaces, then*

$$\prod_{i \in I} X_i$$

is compact.

Exercise 9.

1. Show that an arbitrary product of Hausdorff spaces is Hausdorff.
2. Show that a subspace of a Hausdorff space is Hausdorff.

We will now apply the topology skills we obtained to give a topology to Φ_A . Note that Φ_A is a subset of A^\vee , the vector space of bounded linear functionals from A to \mathbb{C} .

Definition 8.10. Let Y be a Banach space. The *weak** topology on Y^\vee is the weakest topology making all evaluation maps $\text{ev}_y: Y^\vee \rightarrow \mathbb{C}$ continuous for $y \in Y$.

Note that the weak*-topology is simply the topology of pointwise convergence. In other words, we have that $f_n \rightarrow f$ in Y^\vee if and only if $f_n(y) \rightarrow f(y)$ for all $y \in Y$. However, Y^\vee might not be first countable, so we have to be slightly careful in describing the topology only using converging sequences. In general, we would need *nets*, which generalize sequences to larger index sets. In particular, we can't show that a subspace M of Y^\vee is closed by showing that it contains all limits of sequences in M that converge in Y^\vee .

In the operator norm topology, all evaluation maps $\text{ev}_y: Y^\vee \rightarrow \mathbb{C}$ for $y \in Y$ are bounded since

$$|\text{ev}_y(f)| = |f(y)| \leq \|f\| \|y\|$$

for $f \in Y^\vee$. It follows that the weak* topology on Y^\vee is weaker than the operator norm topology. If Y is finite-dimensional, then these two topologies agree, but in general the weak* topology is strictly weaker:

Example 8.11. Let c_0 be the closed subspace of ℓ^∞ consisting of sequences x_n such that $\lim_n x_n = 0$. It can be shown that the map $\ell^1 \rightarrow (c_0)^\vee$ that sends $x_n \in \ell^1$ to the functional

$$a_n \mapsto \sum_{n \in \mathbb{N}} x_n a_n.$$

is an isomorphism of Banach algebras (exercise on sheet 1).

Let $e_n \in \ell^1$ be the sequence that is zero everywhere except equal to 1 at n . Then the sequence $(e_n)_{n \in \mathbb{N}}$ converges to 0 in the weak* topology. However, it does not converge at all in the norm.

Note that the weak* topology is contravariantly functorial: if $T: Y_1 \rightarrow Y_2$ is a bounded linear map, then it defines a linear map $T^\vee: Y_2^\vee \rightarrow Y_1^\vee$ by precomposition. This is well-defined; if $f \in Y_2^\vee$ is a bounded linear functional, then since T is bounded, so is $T^\vee(f) = f \circ T$. One can check that T^\vee is continuous in the weak*-topology; if $y_1 \in Y_1$, then the diagram

$$\begin{array}{ccc} Y_2^\vee & \xrightarrow{T^\vee} & Y_1^\vee \\ & \searrow \text{ev}_{Ty_1} & \swarrow \text{ev}_{y_1} \\ & \mathbb{C} & \end{array}$$

commutes. Therefore any open of the form $\text{ev}_{y_1}^{-1}(U) \subseteq Y_1^\vee$ for some open $U \subseteq \mathbb{C}$ has its preimage $(T^\vee)^{-1} \text{ev}_{y_1}^{-1}(U) = \text{ev}_{Ty_1}^{-1}(U)$ open in Y_2^\vee . It is easy to check that this is contravariantly functorial.

The following theorem is usually proven as a difficult result in functional analysis, but it is not that difficult in case you already believe in Tychonoff's theorem.

Theorem 8.12 (Banach-Alaoglu). *If X is a Banach space, then the unit ball¹¹ in X^\vee is compact Hausdorff in the weak* topology.*

Proof. (Assuming Tychonoff's theorem) Let $B \subseteq X$ be the vectors such that $\|x\| \leq 1$ and similarly $B' \subseteq X^\vee$ for the unit ball in X^\vee . Consider the map

$$B' \rightarrow \prod_{x \in B} D \quad \phi \mapsto (\phi(x))_{x \in B}$$

where $D \subseteq \mathbb{C}$ is the unit disk. This map is well defined, since $\|\phi\| = 1$ implies that $|\phi(x)| \leq \|\phi\| \|x\| = 1$ for all $x \in B$. Also note that it is injective, since ϕ is uniquely determined by its restriction to B . It is easy to see that X^\vee has the subspace topology inside the big product.

Since D is a closed and bounded subset of \mathbb{C} , it is compact Hausdorff and therefore $\prod_{x \in B} D$ is compact Hausdorff. To finish the proof, we will show that the unit ball in X^\vee is closed, after which we will apply Proposition 8.8 to conclude it is also compact Hausdorff.

For a fixed $x \in B$, consider the map

$$\alpha_x: X^\vee \rightarrow \mathbb{R}_{\geq 0} \quad \phi \mapsto |\phi(x)|.$$

This map is weak* continuous, because it is a composition of an evaluation map and the continuous map $\mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ given by absolute value. By definition of the operator norm, the norm map $\|\cdot\|: X^\vee \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$\phi \mapsto \sup_{x \in B} |\phi(x)| = \sup_{x \in B} \alpha_x(\phi).$$

We see that the unit ball in X^\vee is given by

$$\|\cdot\|^{-1}[0, 1] = \{\phi \in X^\vee : \sup_{x \in B} \alpha_x(\phi) \leq 1\} = \bigcap_{x \in B} \{\phi \in X^\vee : \alpha_x(\phi) \leq 1\},$$

which is an intersection of closed sets and therefore closed. □

Now, let A be a commutative Banach algebra. Recall that every algebra homomorphism $A \rightarrow \mathbb{C}$ is continuous and so we can define:

Definition 8.13. The *Gelfand spectrum* is the subset $\Phi_A \subseteq A^\vee$ given the subspace topology, where A^\vee has the weak* topology.

It follows from the fact that both the weak* topology as well as the subspace topology are defined as a weak topology, that we have put the weakest topology on Φ_A making \hat{a} continuous for all $a \in A$.

Remark 8.14. The Zariski spectrum of a general commutative ring R is given by its collection of maximal ideals (often prime ideals are considered instead). The Zariski spectrum also has a topology. It follows by the previous section that for a commutative Banach algebra A , the Zariski spectrum and Gelfand spectrum are in bijection. However, these spaces are in general not homeomorphic, even though they are when A is a C^* -algebra.

¹¹Note that the unit ball is taken with respect to the operator norm on X^\vee . The weak* topology is in general not even metrizable, so it is not clear what 'unit ball in the weak* topology' would even mean.

It follows by the contravariant functoriality of the weak*-topology that

Proposition 8.15. *The Gelfand spectrum defines a contravariant functor from commutative Banach algebras (equipped with continuous homomorphisms as morphisms) to compact Hausdorff spaces.*

Proof. We first prove that $\Phi_A \subseteq A^\vee$ is closed in the weak* topology. For $a, b \in A$ define the map $\lambda_{a,b}: A^\vee \rightarrow \mathbb{C}$ by

$$\lambda_{a,b}(\phi) = \phi(ab) - \phi(a)\phi(b) = \text{ev}_{ab}(\phi) - \text{ev}_a(\phi)\text{ev}_b(\phi).$$

Note that $\lambda_{a,b}$ is weak*-continuous as it is a combination of evaluation maps and continuous operations on \mathbb{C} . It follows that

$$\Phi_A = \bigcap_{a,b \in A} (\lambda_{a,b})^{-1}(\{0\})$$

is a closed set.

Since we have shown that $\|\phi\| = 1$ for every $\phi \in \Phi_A$, it is also contained in the unit ball of A^\vee . It follows from Banach-Alaoglu that Φ_A is compact Hausdorff.

If $\psi: A \rightarrow B$ is a homomorphism of Banach algebras, we define $\Phi_\psi: \Phi_B \rightarrow \Phi_A$ by ψ^\vee . In other words, we map $\phi: B \rightarrow \mathbb{C}$ to $A \xrightarrow{\psi} B \xrightarrow{\phi} \mathbb{C}$, which is again a homomorphism. Checking that this is a functor is an exercise. \square

Remark 8.16. Stupid but important observation: for $\phi_1, \phi_2 \in \Phi_A$ distinct, there exists an $a \in A$ such that $\hat{a}(\phi_1) \neq \hat{a}(\phi_2)$.

What is the relationship between the Gelfand spectrum and the spectrum of an element?

Lemma 8.17. *Let $a \in A$. Then*

$$\text{Spec}(a) = \{\phi(a) : \phi \in \Phi_A\} = \text{Spec}(\hat{a}).$$

Proof. The second equality follows from the general fact that the spectrum of an element in $C(X)$ is its image, so we prove the first equality.

If $\phi \in \Phi_A$, then $\phi(a) \in \text{Spec}(a)$. Indeed, if $a - \phi(a)$ would have an inverse b , then

$$1 = \phi(1) = \phi(b(a - \phi(a))) = \phi(b)(\phi(a) - \phi(a)) = 0,$$

a contradiction.

Conversely, if $\lambda \in \text{Spec}(a)$, then the ideal generated by $a - \lambda$ is proper, which is then contained in some maximal ideal I by Zorn's lemma. Then $\phi: A \rightarrow A/I \cong \mathbb{C}$ satisfies $\phi(a - \lambda) = 0$, which means $\phi(a) = \lambda$. \square

9 Gelfand duality part I (October 2)

Theorem 9.1. *The Gelfand transform defines a homomorphism $A \rightarrow C(\Phi_A)$ of Banach algebras of operator norm 1.*

Proof. The fact that this is a homomorphism of algebras follows immediately from the fact that elements $\phi \in \Phi_A$ are homomorphisms of algebras. Note that by Lemma 8.17 we have

$$\|\hat{a}\| = \sup_{\phi \in \Phi_A} |\hat{a}(\phi)| = \sup_{\phi \in \Phi_A} |\phi(a)| = \sup_{\lambda \in \text{Spec } a} |\lambda|.$$

It follows that the Gelfand transform is of operator norm 1 since

$$\|\hat{a}\| = \sup_{\lambda \in \text{Spec } a} |\lambda| \leq \|a\|.$$

The supremum is also achieved since $\|\hat{1}\| = 1$. □

The Gelfand transform maps closed sets to closed sets. Indeed, if $K \subseteq A$ is closed and $a_n \in K$ is a sequence converging to a , then

$$\|\hat{a}_n - \hat{a}\| = \sup_{\phi \in \Phi_A} |\phi(a_n) - \phi(a)| \leq \|\phi\| \|a_n - a\| = \|a_n - a\| \rightarrow 0$$

and so $\hat{a} \in K$.

9.1 Semisimple and uniform algebras

The spectrum gives no information about nilpotents.

Example 9.2. The algebra of ‘dual numbers’ $A = \mathbb{C}[\epsilon]/(\epsilon^2)$ with $\|a + b\epsilon\| = |a| + |b|$ is a Banach algebra. Indeed, if $a_1 + b_1\epsilon, a_2 + b_2\epsilon \in A$, then

$$\|(a_1 + b_1\epsilon)(a_2 + b_2\epsilon)\| = \|a_1a_2 + (a_1b_2 + b_1a_2)\epsilon\| = |a_1a_2| + |a_1b_2 + b_1a_2|$$

which is smaller or equal to

$$\|(a_1 + b_1\epsilon)\| \|(a_2 + b_2\epsilon)\| = |a_1||a_2| + |a_1||b_2| + |b_1||a_2| + |b_1||b_2|.$$

One can check that the only ideals of A are $\{0\}$ and (ϵ) (exercise). In particular, (ϵ) is the unique maximal ideal and so Φ_A is a single point. So Φ_\bullet cannot distinguish A from \mathbb{C} .

In fact, the Gelfand transform of any nilpotent element will be trivial since

$$\phi(a)^n = \phi(a^n) = 0 \implies \phi(a) = 0.$$

More generally it turns out that the Gelfand transform does not see any ‘quasi-nilpotents’. To define these, recall that if A is an algebra, then any nilpotent $a \in A$ has $\text{Spec } a = \{0\}$.

Definition 9.3. An element $a \in A$ is called a *quasi-nilpotent* if $\text{Spec } a = \{0\}$.

Definition 9.4. The *Jacobson radical* of a commutative ring R is the intersection of all maximal ideals.

Arbitrary intersections of ideals are again ideals, so the Jacobson radical is an ideal. A commutative ring is *semisimple* if its Jacobson radical vanishes.

Theorem 9.5. *The following are equivalent for a commutative Banach algebra A*

1. A is semisimple.
2. The Gelfand transform is injective.
3. There are no quasi-nilpotents.

Proof. Let $a \in A$. We are done by observing the following chain of equivalent statements:

1. a lies in the Jacobson radical of A
2. a lies in every maximal ideal of A
3. $\phi(a) = 0$ for all $\phi \in \Phi_A$
4. $\hat{a} = 0$, i.e. a lies in the kernel of the Gelfand transform
5. $\|\hat{a}\| = \sup_{\lambda \in \text{Spec}(a)} |\lambda| = 0$
6. $\text{Spec}(a) = \{0\}$

□

The *nilradical* N of a ring R is the intersection of all prime ideals. Note that if $x \in R$ is nilpotent, then $x \in N$. Indeed, if P is prime, then $x \cdot x^{n-1} = 0 \in P$ for some n , so either $x \in P$ or $x^{n-1} \in P$. In the latter case it also follows by primality and induction that $x \in P$. It is well-known (see e.g. wikipedia) that the converse holds; the nilradical is the set of all nilpotents of R . Since every maximal ideal is prime, the nilradical is contained in the Jacobson radical.

It follows from the proof of Theorem 9.5 that in a commutative Banach algebra the Jacobson radical exactly consists of the quasi-nilpotents. This is consistent with the fact that every nilpotent is a quasi-nilpotent.

Remark 9.6. There are non-semisimple commutative Banach algebras without any nilpotents, i.e. the nilradical vanishes but the Jacobson radical does not. So these Banach algebras have no nontrivial nilpotents, but they do have nontrivial quasinilpotents. For an example, see the exercise sheet.

Definition 9.7. A commutative Banach algebra is called *uniform* if $\|a^2\| = \|a\|^2$ for all $a \in A$.

It immediately follows from the following remark that any commutative C^* -algebra is uniform.

Remark 9.8. Any normal element $a \in A$ of a not necessarily commutative C^* -algebra satisfies $\|a^2\| = \|a\|^2$ (an element is called *normal* if $a^*a = aa^*$). Indeed,

$$\|a^*\|^2 \|a\|^2 = \|a\|^4 = \|a^*a\|^2 = \|(a^*a)^*(a^*a)\| = \|(a^2)^*a^2\| \leq \|(a^2)^*\| \|a^2\| \leq \|a^*\|^2 \|a^2\|.$$

using that $\|a^*\| = \|a\|$.

Recall that a linear map $T: \mathcal{X} \rightarrow \mathcal{Y}$ is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$. Note that this is strictly stronger than $\|T\| = 1$.

Proposition 9.9. *The following are equivalent for a commutative Banach algebra A .*

1. The Gelfand transform is an isometry.
2. A is uniform.

In particular, every uniform algebra is semisimple.

Proof. By the spectral radius formula, the Gelfand transform is an isometry if and only if

$$\|a\| = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} \quad (5)$$

for all $a \in A$. Since in particular, the right hand side always converges, the subsequence $\|a^{2^k}\|^{1/2^k}$ also converges.

Suppose A is uniform, so that $\|a^{2^k}\|^{1/2^k} = \|a\|$ for all k . Then the sequence $\|a^{2^k}\|^{1/2^k}$ converges to $\|a\|$ as $k \rightarrow \infty$ and so we obtain 5.

Conversely, suppose that $\|a^2\| \neq \|a\|^2$ for some element $a \in A$. By the Banach algebra property we can't have that $\|a^2\| > \|a\|^2$. Then the fact that $\|a^2\|^{1/2} < \|a\| - \epsilon$ for some small enough $\epsilon > 0$ implies that $\|a^{2^k}\|^{1/2^k} \leq \|a^2\|^{1/2} < \|a\| - \epsilon$ for all k and hence $\lim_{k \rightarrow \infty} \|a^{2^k}\|^{1/2^k} \leq \|a\| - \epsilon < \|a\|$. So Equation 5 does not hold.

Since every isometry is an injection, it follows from Theorem 9.5 that A is semisimple. \square

Remark 9.10. An example of a uniform algebra which is not a C^* -algebra is the disk algebra. We will consider this Banach algebra in the exercises.

Remark 9.11. Let $A \subseteq C(X)$ be a closed subalgebra for some compact space X . Then A is clearly a uniform algebra. Conversely, if A is a uniform algebra, then it follows by Proposition 9.9 that $A \subseteq C(\Phi_A)$ is a closed subalgebra with the norm of A induced by $C(\Phi_A)$. More generally, if A is any commutative Banach algebra, then its image in $C(\Phi_A)$ is a uniform algebra.

9.2 The Gelfand transform for C^* -algebras

We will show that the Gelfand transform gives an isomorphism $A \cong C(\Phi_A)$ if A is a commutative C^* -algebra.

Proposition 9.12. *Any commutative C^* -algebra A is uniform (hence semisimple) and the Gelfand transform is a $*$ -homomorphism.*

Proof. We first prove that the Gelfand transform preserves self adjoint elements. Let $a \in A$ be self adjoint. Then since $\text{Spec } a \subseteq \mathbb{R}$, we have that \hat{a} only takes real values. It follows that $\widehat{\hat{a}} = \hat{a} = \widehat{a^*}$.

To show that the Gelfand transform preserves the $*$, we let $b \in A$ and write it as a sum $b = \Re b + i\Im b$, where $\Re b$ and $\Im b$ are self adjoint. It thus follows that

$$\widehat{(b^*)} = \widehat{\Re b - i\Im b} = \widehat{\Re b} - i\widehat{\Im b} = \widehat{\Re b}^* - i\widehat{\Im b}^* = (\widehat{\Re b} + i\widehat{\Im b})^* = \widehat{(b)}^*.$$

(we have essentially proven that a homomorphism that preserves self adjoint elements is a $*$ -homomorphism)

Finally, we obtain that A is uniform since

$$\|a\|^2 = \|a^*a\| = \text{Spec}_{\lambda \in \text{Spec}(a^*a)} |\lambda| = \|\widehat{a^*a}\| = \|\widehat{a^*}\widehat{a}\| = \|\widehat{a}\|^2.$$

In particular the Gelfand transform is injective and so A is semisimple. \square

Note the consequence of the above result that every algebra homomorphism $A \rightarrow \mathbb{C}$ out of a C^* -algebra is a $*$ -homomorphism. It therefore follows that every maximal ideal I is closed under the $*$ -operation, i.e. $a \in I$ implies $a^* \in I$. This is more generally true for any closed ideal in a not necessarily commutative C^* -algebra, but we won't prove that here.

For a commutative C^* -algebra A , the surjectivity of the Gelfand transform

$$A \rightarrow C(\Phi_A)$$

is not easy to prove. However, it is an easy corollary of an important theorem:

Theorem 9.13 (Stone-Weierstraß). *Let X be a compact Hausdorff space and let $A \subseteq C(X)$ be a subalgebra which separates points, i.e. for all distinct $x, y \in X$ there exists an $a \in A$ such that $a(x) \neq a(y)$. If A is closed under the $*$ operation of $C(X)$, then A is dense.*

Corollary 9.14. *Let A be a commutative C^* -algebra. Then the Gelfand transform*

$$A \rightarrow C(\Phi_A)$$

is a $$ -isomorphism.*

Proof. It follows from Proposition 9.12 that the Gelfand transform is an injective $*$ -homomorphism. The image of A under the Gelfand transform is an algebra closed under $*$, which clearly separates points. It follows by the Stone-Weierstraß theorem that A is dense in $C(\Phi_A)$. However, A is closed in $C(\Phi_A)$; A is complete and the Gelfand transform maps closed sets to closed sets. \square

10 Gelfand duality II (October 7)

We are now arriving at the main goal of the first part of this course: We want to show that there is an equivalence of categories $cC^*Alg \simeq CptHaus^{op}$. Here cC^*Alg is the category in which objects are commutative C^* -algebras and morphisms are $*$ -homomorphisms, and $CptHaus$ is the category in which objects are topological spaces which are both compact and Hausdorff.

Recall from Section 3 what an equivalence of categories is and what the superscript op means (opposite category), as we are now going to unpack this. First of all, we will need a functor $F: CptHaus^{op} \rightarrow cC^*Alg$. We take it to be $X \mapsto C(X)$ on objects, which as we discussed extends to a contravariant functor via $F(g)(f) = f \circ g$ for $f \in C(X)$ and $g: X \rightarrow Y$ a continuous map. In the other direction, we have seen the functor $G: cC^*Alg \rightarrow CptHaus^{op}$ given by the Gelfand spectrum $A \mapsto \Phi_A$ on objects and $G(\psi)(\phi) = \phi \circ \psi$ on morphisms.

We now want to show these are inverses, i.e. ' $X = \Phi_{C(X)}$ ' and ' $A = C(\Phi_A)$ '. These shouldn't be equalities, but instead natural isomorphisms. For the latter we already have a candidate; we have seen that for a commutative C^* -algebra, the Gelfand transform

$$A \rightarrow C(\Phi_A)$$

is an isomorphism. The fact that this isomorphism is natural in $*$ -homomorphisms is completely tautological but also somewhat mind bending:

Lemma 10.1. *The Gelfand transform defines a natural transformation from id_{cC^*Alg} (the identity functor on the category of commutative C^* -algebras) to FG .*

Proof. If $\psi: A \rightarrow B$ is a $*$ -homomorphism, then we have to show that the diagram

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto \hat{a}} & C(\Phi_A) \\ \downarrow \psi & & \downarrow FG(\psi) \\ B & \xrightarrow{b \mapsto \hat{b}} & C(\Phi_B) \end{array}$$

commutes. This follows by direct computation

$$[FG\psi](\hat{a})(\phi) = [\hat{a} \circ G(\psi)](\phi) = \hat{a}(\phi \circ \psi) = \phi(\psi(a)) = \widehat{\psi(a)}(\phi)$$

where $a \in A$ and $\phi \in \Phi_B$. □

The natural transformation in the other direction we haven't discussed much yet. For a compact Hausdorff space X it is given by the 'evaluation map' $\text{ev}: X \rightarrow \Phi_{C(X)}$. It sends $x \in X$ to the element $\text{ev}_x \in \Phi_{C(X)}$ corresponding to the 'evaluating at x ' homomorphism $\text{ev}_x: C(X) \rightarrow \mathbb{C}$.

This map is continuous for the usual weak* topology on $\Phi_{C(X)}$; if $f \in C(X)$ and $U \subseteq \mathbb{C}$, then

$$(\text{ev}_f)^{-1}(U) = \{\phi \in \Phi_{C(X)} : \phi(f) \in U\} \subseteq \Phi_{C(X)}$$

is open and it suffices to check that its preimage under $X \rightarrow \Phi_{C(X)}$ is open. But its preimage is

$$\{x \in X : \text{ev}_x(f) \in U\} = f^{-1}(U),$$

which is open by continuity of f .

Lemma 10.2. *The evaluation map defines a natural transformation from $\text{id}_{CHaus^{op}}$ (the identity functor compact Hausdorff spaces) to GF .*

Proof. The proof is entirely analogous: if $g: X \rightarrow Y$ is a continuous map, we have to show that the diagram

$$\begin{array}{ccc} X & \longrightarrow & \Phi_{C(X)} \\ \downarrow g & & \downarrow GF(g) \\ Y & \longrightarrow & \Phi_{C(Y)} \end{array}$$

commutes. If $f \in C(Y)$ and $x \in X$ then

$$[GFg](\text{ev}_x)(f) = [\text{ev}_x \circ Fg](f) = \text{ev}_x(Fg(f)) = \text{ev}_x(f \circ g) = \text{ev}_{g(y)}(f)$$

□

To finish the proof of Gelfand duality, we have to show that $X \rightarrow \overline{\Phi_{C(X)}}$ is a homeomorphism if X is compact Hausdorff. Note that it suffices to show that the map is bijective:

Lemma 10.3. *If $g: X \rightarrow Y$ is a continuous bijection where X is compact and Y is Hausdorff, then g is a homeomorphism.*

Proof. It suffices to show that g maps closed sets to closed sets. Let $K \subseteq X$ be closed. Any closed subset of a compact set is closed, and so K is compact. The image of a compact set under a continuous map is compact, and so $f(K)$ is compact. A compact subset of a Hausdorff space is closed and so $f(K)$ is closed. □

So when is $X \rightarrow \overline{\Phi_{C(X)}}$ injective? This is saying that for all $x, y \in X$ distinct, $\text{ev}_x \neq \text{ev}_y$. In other words, there exists $f \in C(X)$ such that $f(x) \neq f(y)$. Note that without loss of generality, we can take f to be valued in $[0, 1]$, $f(x) = 0$, and $f(y) = 1$. Indeed, we can construct a continuous map $\mu: \mathbb{C} \rightarrow [0, 1]$ with the property that $f(x) \mapsto 0$ and $f(y) \mapsto 1$. For example, take $\mu(z)$ to be $\max(1, d(z, f(x)))$, where $d(z, f(x))$ is the distance from z to $f(x)$ in the complex plane normalized so that the distance between $f(x)$ and $f(y)$ is 1, i.e. $d(z, f(x)) = \frac{|z - f(x)|}{|f(y) - f(x)|}$. In other words, X needs to be completely Hausdorff:

Definition 10.4.

- A topological space X is called *completely Hausdorff* if for any two distinct points $x, y \in X$ there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.
- A topological space is called T_1 if all singleton sets are closed.
- A topological space is called *normal* if closed sets can be separated by open neighbourhoods.
- A topological space is called T_4 if it is T_1 and normal.

Note that T_4 implies Hausdorff.

Separation axioms fall in two types:

1. separate two things by open sets
2. separate two things by a continuous real-valued function

It is usually not difficult to find implications from the second point to the first. For example, note that completely Hausdorff implies Hausdorff. The other direction usually requires difficult proofs. One such is Urysohn's lemma, which we will not prove.

Theorem 10.5 (Urysohn). *A topological space is normal if and only if closed sets can be separated by continuous functions. More precisely, if $A, B \subseteq X$ are disjoint closed sets, then there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

In particular, it follows that T_4 implies completely Hausdorff. For general topological spaces, all these implications

$$T_4 \implies \text{completely Hausdorff} \implies \text{Hausdorff} \implies T_1$$

are strict. However, it turns out that compact Hausdorff spaces are already normal. In particular, it follows that in that case $X \rightarrow \Phi_{C(X)}$ is injective. For those who want to learn more, sections 31,32, and 33 of Munkres' topology book [4] are an excellent reference.

Theorem 10.6. *$X \rightarrow \Phi_{C(X)}$ is surjective and hence a homeomorphism.*

Proof. Let $\phi: C(X) \rightarrow \mathbb{C}$ be an element of $\Phi_{C(X)}$. Our goal is to find an $x \in X$ so that $\text{ev}_x = \phi$. Consider the set

$$Y = \{x \in X : f(x) = 0 \forall f \in C(X) \text{ with } \phi(f) = 0\}.$$

The claim is that if $Y \neq \emptyset$, then we are done. Indeed, if $y \in Y$, then $\text{ev}_y(f) = f(y) = 0$ whenever $\phi(f) = 0$ by definition of Y . But that means that $\ker \phi \subseteq \ker \text{ev}_y$. However, since $\ker \phi$ is a maximal ideal and $\ker \text{ev}_y$ is a proper ideal, this shows $\ker \phi = \ker \text{ev}_y$ and hence $\phi = \text{ev}_y$.

So suppose for a contradiction that $Y = \emptyset$. Our goal will be to show that $\phi(1) = 0$, contradicting the fact that ϕ is a homomorphism. For every $x \in X$, we know that $x \notin Y$, and so we can pick some function $f_x \in C(X)$ with the property that $f_x(x) \neq 0$ but $\phi(f_x) = 0$. Consider the open sets

$$U_x = \{x' \in X : f_x(x') \neq 0\}.$$

Since $x \in U_x$ for all $x \in X$ by our choice of f_x , this is an open cover of X . Since X is compact, there is a finite subcover U_{x_1}, \dots, U_{x_n} . But this means that

$$|f_1|^2 + \dots + |f_n|^2 = f_1 f_1^* + \dots + f_n f_n^*$$

is a nowhere vanishing function on X . Also note that this function is still in the kernel of ϕ because ϕ is a *-homomorphism. But a nowhere vanishing function is invertible in the algebra $C(X)$ and hence we see that $1 \in \ker \phi$. \square

We have therefore proven Gelfand duality

Theorem 10.7. *There is an equivalence of categories*

$$\text{CptHaus}^{\text{op}} \simeq \text{c}C^* \text{Alg}$$

11 Review lecture (October 9)

11.1 Hilbert spaces: bases, separability and orthogonal complements

We went a bit fast when we were doing Hilbert spaces before, so let's do a review and emphasize some subtleties.

The usual notion of (Hamel) basis is not very useful for infinite-dimensional topological vector spaces. We instead want to require that the closure of the linear span of linearly independent vectors gives the whole vector space. Whenever we talk about bases of Hilbert spaces, we will always mean this variant.

Definition 11.1. We say that $\{e_i\}_{i \in I}$ is an *orthonormal basis* of \mathcal{H} if $\langle e_i, e_j \rangle = \delta_{i,j}$ and

$$\overline{\text{span}\{e_i : i \in I\}} = \mathcal{H}.$$

Example 11.2. The collection of sequences $(\delta_{r,s})_{s \in S}$ for $r \in S$ forms an orthonormal basis of $\ell^2(S)$.

Once you have a basis you can work with 'infinite by infinite matrices'. Let's do this for a countable basis. Let $E_{i,j} : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ be the elementary matrix that sends $e_j \mapsto e_i$ and $e_k \mapsto 0$ if $k \neq j$. In other words, $E_{i,j}e_k = \delta_{j,k}e_i$. This extends by linearity to a linear operator on the span of the e_i , i.e. those sequences with finitely many nonzero entries. If $E_{i,j}$ has a chance of being continuous/bounded on all of $\ell^2(\mathbb{N})$ then it is uniquely determined on the e_i because their span is dense. Moreover, $E_{i,j}$ is then given by

$$E_{i,j}(x_1, x_2, \dots) = (0, \dots, 0, x_j, 0 \dots),$$

where the x_j is put on the i th spot. This is indeed bounded because

$$\|E_{i,j}x\| = |x_j| \leq \sqrt{\sum_{n=0}^{\infty} |x_n|^2} = \|x\|$$

We have $E_{i,j}^* = E_{j,i}$. Indeed,

$$\langle e_k, E_{i,j}e_l \rangle = \langle e_k, \delta_{j,l}e_i \rangle = \delta_{j,l}\delta_{k,i} \quad \langle E_{j,i}e_k, e_l \rangle = \langle \delta_{i,k}e_j, e_l \rangle = \delta_{i,k}\delta_{j,l}$$

which is the same. It suffices to show this on the basis and then extend by linearity and continuity.

Example 11.3. Let $S : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ be the Hilbert hotel operator (usually called shift operator)

$$(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots).$$

Note that S is not surjective, but it is injective. This is interesting; for a finite-dimensional vector space V , it would follow by the rank nullity theorem that any injective linear map $T : V \rightarrow V$ is automatically surjective. However, the rank nullity theorem for Hilbert spaces says that

$$\dim \text{Im } T + \dim \ker T = \dim \mathcal{H}$$

if $T : \mathcal{H} \rightarrow \mathcal{H}$. Here both sides can be ∞ and we define $\infty + n = \infty + \infty = \infty$ for all integers n . We saw in this example that if $\dim \text{Im } T = \dim \mathcal{H} = \infty$, $\text{Im } T$ can still be nonzero (in fact, it can even be infinite dimensional!).

The adjoint of S is given by

$$(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots).$$

Note that $S^*S = \text{id}$ but $SS^* \neq \text{id}$ so S is not unitary. In fact SS^* is not even invertible, so this is an example of a, b in a C^* -algebra for which $\text{Spec}(ab) \neq \text{Spec}(ba)$. However, it turns out that we always have $\text{Spec}(ab) \setminus \{0\} = \text{Spec}(ba) \setminus \{0\}$.¹²

Observe that S has no eigenvalues whatsoever since

$$(0, x_1, x_2, \dots) = \lambda(x_1, x_2, \dots)$$

would imply $x_1 = 0$, but then $x_2 = 0, \dots$ but then $x_i = 0$ for all i . On the other hand, we know that $\text{Spec } S \neq \emptyset$. In fact, it turns out that $\text{Spec}(S)$ is the closed unit disk of radius 1.

Theorem 11.4. *Every Hilbert space \mathcal{H} admits an orthonormal basis $\{e_i\}_{i \in I}$ for some set I . This defines a (linear bounded) isomorphism $\mathcal{H} \cong \ell^2(I)$ and hence the isomorphism class of a Hilbert space is determined by the cardinality of I .*

Proof. See for example Chapter 6 of the book [3]. □

¹²For people knowing quantum mechanics, this is bad: you want to have operators p, x such that $[p, x] = i\hbar$. But this is not possible in a C^* -algebra

12 Projections (October 16)

Definition 12.1. Let R be a commutative ring. An *idempotent* is an element $e \in R$ such that $e^2 = e$. A *projection* in a C^* -algebra A is a self adjoint idempotent, i.e. an element $p \in A$ such that $p^2 = p$ and $p^* = p$.

The trivial projections are 0 and 1.

Example 12.2. A projection in $C(X)$ is a continuous map $f: X \rightarrow \{0, 1\} \subseteq \mathbb{R}$. Since $\{0, 1\}$ has the discrete topology, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are clopen. We see that a projector is the same as the choice of a clopen set in X . In particular, $C(X)$ has no nontrivial projections if and only if X is connected. Note that in this example, the condition that f is self adjoint is redundant; any idempotent is a projection. This example vastly generalizes to a correspondence between idempotents in a commutative ring R and clopens in the Zariski spectrum of R . This is a nontrivial theorem, see e.g. [1, Theorem 2].

Example 12.3. Let $a \in A$ satisfy $a^*a = 1$. Then aa^* is a projection. Indeed, $(aa^*)^* = a^{**}a^* = aa^*$ and $a(a^*a)a^* = aa^*$ by assumption. The projection is trivial if and only if a is unitary. As a subexample, one can take a to be the Hilbert hotel operator on $B(\ell^2(\mathbb{N}))$. The resulting projection projects onto the sequences with the first entries zero. We will see later in this lecture the relationship between projections in $B(\mathcal{H})$ and projections onto subspaces.

Lemma 12.4. Let $e \in R$ be idempotent. Then $\text{Spec}(e) \subseteq \{0, 1\}$.

Proof. If $\lambda \neq 0, 1$, a computation shows that a two-sided inverse of $e - \lambda$ is

$$\left(\frac{e}{\lambda(1-\lambda)} - \frac{1}{\lambda} \right).$$

□

Corollary 12.5. Any projection $p \in A$ in a C^* -algebra is positive.

It is true that an element $a \in A$ of a C^* -algebra A is positive if and only if $a = b^*b$ for some $b \in A$ but we haven't proven this yet. Note that $p = p^*p$ and so we would have another proof that p is positive.

Recall that it follows from the spectral radius formula that if $a \in A$ is self-adjoint and $\text{Spec } a = \{\lambda\}$, then $a = \lambda$. We see that a nontrivial projection has $\text{Spec } p = \{0, 1\}$.

Definition 12.6. Two projections $p, p' \in A$ are called *orthogonal* if $pp' = 0$.

Note that if $p, p' \in A$ are orthogonal, then also $p'p = 0$ since

$$0 = (pp')^* = (p')^*p^* = p'p.$$

It now follows by a short computation that $p+p'$ is a projection. Note that $p+p' \geq p$ and $p+p' \geq p'$.

If $p \in A$ is a projection, then its *orthocomplement* $1-p \in A$ is a projection orthogonal to p . We have that $p \leq 1$ and $p \geq 0$ for every projection $p \in A$, so the poset of projections has a greatest and smallest element. Note that $p \mapsto 1-p$ is an order-reversing bijection on the collection of projections. It follows in particular that the poset admits suprema if and only if it admits infima.

We will now study projections in Hilbert spaces. They turn out to be exactly projections onto closed subspaces. We will not prove this, but we will give some intuition why it should be true. For this, we will need to know about orthogonality.

Two sets S_1, S_2 in a Hilbert space are called *orthogonal* if all inner products between elements of them are zero, i.e. $\langle x_1, x_2 \rangle = 0$ for all $x_1 \in S_1$ and $x_2 \in S_2$. Note that if two sets are orthogonal, then their intersection can at most contain $0 \in \mathcal{H}$. In particular, if $C_1, C_2 \subseteq \mathcal{H}$ are orthogonal subspaces, they are linearly independent. They therefore form an internal orthogonal direct sum. So if for example C_1 and C_2 together span \mathcal{H} , then there is an isomorphism of Hilbert spaces $C_1 \oplus C_2 \rightarrow \mathcal{H}$ given by $(x_1, x_2) \mapsto x_1 + x_2$. Note that it is possible that $C_1 \cap C_2 = \{0\}$ but C_1 and C_2 are not orthogonal. In that case the above isomorphism would only be an isomorphism of vector spaces. In other words, the direct sum would only be an algebraic direct sum, not a Hilbert direct sum.

Suppose now that $p, p' \in B(\mathcal{H})$ are orthogonal. Then their images are orthogonal since

$$\langle px, p'y \rangle = \langle x, pp'y \rangle = 0$$

for all $x, y \in \mathcal{H}$. In particular, the images of p and $1 - p$ are orthogonal. Also note that the image of $1 - p$ is exactly $\ker p$; if $px = 0$, then $x = (1 - p)x + px = (1 - p)x$ and the other direction is obvious. We conclude that $\ker p$ and $\text{Im } p$ are orthogonal in \mathcal{H} and span \mathcal{H} . In other words,

$$\mathcal{H} \cong \ker p \oplus \text{Im } p$$

and $p: \mathcal{H} \rightarrow \mathcal{H}$ is given by projecting onto the second factor and including back into \mathcal{H} .

The converse is more difficult, but it turns out that if $C \subseteq \mathcal{H}$ is a closed subspace, there exists a projection onto C . Let $C \subseteq \mathcal{H}$ be a linear subspace.

Definition 12.7. The *orthogonal complement* of C is

$$C^\perp := \{x \in \mathcal{H} : \langle x, y \rangle = 0 \forall y \in C\}.$$

It is straightforward to check that $C^\perp \subseteq \mathcal{H}$ is a subspace and that $C \cap C^\perp = \{0\}$.

Lemma 12.8. *The subspace C^\perp is closed.*

Proof. Let x_n be a sequence in C^\perp converging to $x \in \mathcal{H}$. Let $y \in \mathcal{H}$. We need to show that $\langle x, y \rangle = 0$. This is obvious when $y = 0$, so we can assume $\|y\| \neq 0$. Let $\epsilon > 0$. Pick n so large that $\|x - x_n\| < \epsilon/\|y\|$. Then using Cauchy-Schwarz we get

$$\begin{aligned} |\langle x, y \rangle| &= |\langle x - x_n, y \rangle + \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \\ &\leq \|x - x_n\| \|y\| < \epsilon. \end{aligned}$$

□

Let $C \subseteq \mathcal{H}$ be a subspace. We say that $y \in C$ is a projection of $x \in \mathcal{H}$ on C if y is one of the elements of C closest to x . More precisely,

$$\|x - y\| = \inf\{\|x - y'\| : y' \in C\}$$

Example 12.9. If $x \in C$, then x is a projection of x on C .

Exercise 10. Let $v \in \mathcal{H}$ have norm 1.

1. show that $\langle v, x \rangle v$ is a projection of x on the span of v .
2. show that it is the only projection of x on the span of v .

Note that if $y \in C$ is a projection of x on C , then y is a projection of x on $\text{span}\{y\}$. It follows from the previous exercise that y is unique if it exists. It also follows that if $y' \in C$, then $\langle y', x \rangle = \langle y, x \rangle$.

Projections onto nonclosed subspaces need not exist:

Exercise 11. Let $\ell^2 = \bigoplus_{n \in \mathbb{N}} \mathbb{C}$ be the Hilbert space of square summable sequences and let $c_{00} = \bigoplus_{n \in \mathbb{N}}^{alg} \mathbb{C} \subseteq \ell^2$ be the subspace of eventually zero sequences. Show that a sequence $x \in \ell^2$ admits a projection on C if and only if $x \in c_{00}$.

The following theorem contains much of the nontrivial content of projections. One proof is given in [6, Number 5 Paragraph VII].

Theorem 12.10. *If $C \subseteq \mathcal{H}$ is closed, then every $x \in \mathcal{H}$ has a projection on C .*

Corollary 12.11. *If $C \subseteq \mathcal{H}$ is closed, then $\mathcal{H} \cong C \oplus C^\perp$. Moreover, if we map $x \in \mathcal{H}$ to C by projecting onto the first summand in this direct sum decomposition, we obtain the unique projection of x on C .*

Proof. Since $C \cap C^\perp = \{0\}$ orthogonally, we only need to show that C and C^\perp span \mathcal{H} . Given $x \in \mathcal{H}$, let $y \in C$ be the projection of x on C . If we write $x = (x - y) + y$, then $y \in C$ and $x - y \in C^\perp$. \square

We in particular obtain a projection $p \in B(\mathcal{H})$ with image C by projecting onto the first sum in the above direct sum decomposition and then including it back into \mathcal{H} .

We will be interested in the poset of projections.

Example 12.12. Recall that if A is commutative, then $A \cong C(X)$ and the poset of projections is $\text{clopens}(X)$. In fact, it is not difficult to check that the partial ordering on the projections corresponds to inclusions of clopens.

Every pair of projections p_1, p_2 has a supremum $p_1 \vee p_2$ and an infimum $p_1 \wedge p_2$. On the algebra side, this follows because the minimum (or maximum) of two continuous functions is again continuous, and the minimum (or maximum) of two projections is a projection. On the topology side, it follows from the fact that finite intersections and unions of clopen sets are again clopen.

However, this poset in general does not have infinite suprema. The problem is that if U_i is a family of clopen sets, then $\bigcup_i U_i$ is open but need not be closed and $\bigcap_i U_i$ is closed but need not be open. One counterexample is the space $\mathbb{N} \cup \{\infty\}$, which we will look at in more detail in later lectures.

Example 12.13. We have discussed how the set of projections in $B(\mathcal{H})$ is the set of closed subspaces of C . It is not immediately clear, but it turns out that the partial order corresponds to taking subspaces. In particular, arbitrary infima of projections in $B(\mathcal{H})$ exist; given a family C_i of closed subspaces, $\bigcap_i C_i$ is a closed subspace. It is clearly the largest subspace contained in all C_i . We also obtain that projections in $B(\mathcal{H})$ have all suprema.

13 Von Neumann subalgebras and topologies on $B(\mathcal{H})$ (October 21)

13.1 Commutants

Let A be an algebra and $S \subseteq A$ a subset.

Definition 13.1. The *commutant* of S in A is

$$S' = \{a \in A : as = sa \forall s \in S\}.$$

The *double commutant* of S is $S'' := (S')'$, the commutant of the commutant of S .

Remark 13.2. The commutant of S strongly depends on the algebra A it is embedded in. Therefore the prime notation can be slightly abusive. We also usually just say ‘commutant of S ’ if A is clear from the context.

Note that if $S = A$, then $S' = Z(A)$ is the *center* of A . To be consistent with the terminology in group theory and avoid confusion with the *commutator* it would perhaps be better to call S' the centralizer of S . I will however stick to the standard terminology.

Some basic properties of commutators are:

1. If $S_1 \subseteq S_2$, then $S_2' \subseteq S_1'$. Indeed, if $as_2 = s_2a$ for all $s_2 \in S_2$, then this holds in particular if $s_2 \in S_1$.
2. $S \subseteq S''$: if $s \in S$ and $a \in S'$, then $sa = as$ and so $s \in S''$.
3. Since $S \subseteq S''$ by point 2, we get $S''' \subseteq S'$ by point 1. On the other hand, also $S' \subseteq S'''$ by point 2 applied to S' . It follows that $S' = S^{\text{odd number of '}}$ and $S'' = S^{\text{positive even number of '}}$.

However, in general we have that $S'' \neq S$. For example, note that for any set S , the commutant $S' \subseteq A$ is automatically a subalgebra. So a necessary condition for $S'' = S$ is that S is a subalgebra of A .

In these lectures, we will only be interested in the special case where $A = B(\mathcal{H})$, where \mathcal{H} is a Hilbert space.

13.2 Von Neumann subalgebras

Let $A \subseteq B(\mathcal{H})$ be a subalgebra. Our current goal is to prove the double commutant theorem, which relates the question of whether $A'' = A$ to a topological question in $B(\mathcal{H})$.

For this, suppose that $A \subseteq B(\mathcal{H})$ is a $*$ -subalgebra. By this we mean that $a \in A$ implies that $a^* \in A$. In that case, $A' \subseteq B(\mathcal{H})$ is also a $*$ -subalgebra. Indeed, if $b \in A'$, then $b^*a = (a^*b)^* = (ba^*)^* = ab^*$ for all $a \in A$. Here we used that b and $a^* \in A$ commute.

Note that at this point A need not be a sub C^* -algebra as it need not be closed in the operator norm topology of $B(\mathcal{H})$. We make the following purely algebraic definition:

Definition 13.3. A sub $*$ -algebra $A \subseteq B(\mathcal{H})$ is a *von Neumann subalgebra* if $A'' = A$.

Recall that $A \subseteq A''$, so A is a von Neumann subalgebra if and only if $A'' \subseteq A$.

Example 13.4. For any sub $*$ -algebra $A \subseteq B(\mathcal{H})$, the commutant A' in $B(\mathcal{H})$ is a von Neumann subalgebra. For example, we could take A to be a sub C^* -algebra.

Example 13.5. Let $A = \mathbb{C} \hookrightarrow B(\mathcal{H})$ be given by multiples of the identity. Since every bounded operator commutes with scalar multiplication, we have that $\mathcal{C}' = B(\mathcal{H})$. So $B(\mathcal{H})$ is a von Neumann subalgebra of $B(\mathcal{H})$

Example 13.6. $A = B(\mathcal{H}) \subseteq B(\mathcal{H})$ satisfies $B(\mathcal{H})' = \mathbb{C}$. In other words, the only operators that commutes with all other operators are scalar multiples of the identity. This is not immediately obvious, but is part of the exercise sheet.

Example 13.7. Consider the block diagonal embedding $B(\mathcal{H}) \rightarrow B(\mathcal{H} \oplus \mathcal{H})$ given by

$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

We will say more about block diagonal embeddings later in the lecture when we talk about the ultraweak topology. For now, note that $B(\mathcal{H})' = M_2(\mathbb{C})$ ¹³, where the latter is embedded in $B(\mathcal{H} \oplus \mathcal{H})$ via operators of the form

$$\begin{pmatrix} a \text{id}_{\mathcal{H}} & b \text{id}_{\mathcal{H}} \\ c \text{id}_{\mathcal{H}} & d \text{id}_{\mathcal{H}} \end{pmatrix}.$$

This follows by the previous example and a matrix computation:

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} T_{11}T & T_{12}T \\ T_{21}T & T_{22}T \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} TT_{11} & TT_{12} \\ TT_{21} & TT_{22} \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

The equality holds if and only if $T_{ij} \in B(\mathcal{H})' = \mathbb{C}$ for all $i, j \in \{1, 2\}$.

13.3 The weak operator topology

Unfortunately, when studying von Neumann algebras, it is important to keep track of multiple topologies on $B(\mathcal{H})$. In this lecture, we will introduce two topologies on $B(\mathcal{H})$ different from the operator norm topology we have studied so much: the weak operator topology and the ultraweak operator topology

Definition 13.8. The *weak operator topology (WOT)* on $B(\mathcal{H})$ is the weak topology corresponding to the family of maps

$$\{f_{v,w} : B(\mathcal{H}) \rightarrow \mathbb{C} : v, w \in \mathcal{H}\},$$

where $f_{v,w}(T) = \langle v, Tw \rangle$.

In other words, for every $v, w \in \mathcal{H}$, the maps $f_{v,w}$ are continuous in the WOT and the WOT is the weakest topology making that happen. This means that the opens in $B(\mathcal{H})$ are generated by inverse images of open subsets of \mathbb{C} under the $f_{v,w}$. Recall that a basis of opens in \mathbb{C} is given by open disks of arbitrary radius around an arbitrary point. So the ‘basic’ open neighborhoods of $0 \in B(\mathcal{H})$ are of the form $f_{v,w}^{-1}(B_\epsilon(0))$, where $B_\epsilon(0) \subseteq \mathbb{C}$ is the open disk of radius $\epsilon > 0$ around the origin. Spelling this out, we obtain that the sets

$$f_{v,w}^{-1}(B_\epsilon(0)) = \{T \in B(\mathcal{H}) : |\langle v, Tw \rangle| < \epsilon\}$$

are open. The reason that I put ‘basic’ in quotes is that these sets do not form a basis of the WOT. Indeed, an intersection of two subsets of the form $f_{v,w}^{-1}(B_\epsilon(0))$ for some $v, w \in \mathcal{H}$ need not be of that

¹³In the last exercise we have shown that $B(\mathcal{H})' = \mathbb{C}$ but now we are showing that $B(\mathcal{H})' = M_2(\mathbb{C})$ which can be confusing. The point is that the commutant depends on the embedding, see Remark 13.2

form again. So instead they only form a subbasis, which means that an arbitrary open neighborhood of 0 in $B(\mathcal{H})$ is an arbitrary union of a finite intersection of sets of the form $f_{v,w}^{-1}(B_\epsilon(0))$.

More generally, we could take inverse images of open disks $B_\epsilon(z)$ around a different point $z \in \mathbb{C}$ and obtain

$$\{T \in B(\mathcal{H}) : |\langle v, Tw \rangle - z| < \epsilon\}.$$

In particular, the open neighborhoods around an arbitrary $T_0 \in B(\mathcal{H})$ are generated by

$$\{T \in B(\mathcal{H}) : |\langle v, Tw \rangle - \langle v, T_0 w \rangle| < \epsilon\} = \{T \in B(\mathcal{H}) : |\langle v, (T - T_0)(w) \rangle| < \epsilon\}.$$

Remark 13.9. The weak operator topology on $B(\mathcal{H})$ is weaker than the operator norm topology which we used to make $B(\mathcal{H})$ into a C^* -algebra. This follows from the fact that the $f_{v,w}$ are all bounded linear functionals. Indeed, note that we have $f_{v,w}(\lambda T_1 + T_2) = \lambda f_{v,w}(T_1) + f_{v,w}(T_2)$ and for the continuity,

$$|f_{v,w}(T)| \leq \|v\| \|Tw\| \leq \|T\| \|v\| \|w\|$$

by the Cauchy-Schwarz inequality.

We will soon see that the WOT is in fact strictly weaker than (in other words, not equal to) the operator norm topology when \mathcal{H} is infinite dimensional. When \mathcal{H} is finite-dimensional, the weak operator topology agrees with the operator norm topology.

Example 13.10. The addition map $+: B(\mathcal{H}) \times B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is WOT-continuous. To show this, consider the diagram

$$\begin{array}{ccc} B(\mathcal{H}) \times B(\mathcal{H}) & \xrightarrow{+} & B(\mathcal{H}) \\ \downarrow (f_{v,w}, f_{v,w}) & & \downarrow f_{v,w} \\ \mathbb{C} \times \mathbb{C} & \xrightarrow{+} & \mathbb{C} \end{array}$$

It follows from the fact that $f_{v,w}: B(\mathcal{H}) \rightarrow \mathbb{C}$ is linear that this diagram commutes. Since addition in \mathbb{C} is continuous, going through the diagram using the western route defines a continuous map. Hence the inverse image under $+$ of any basic open $f_{v,w}^{-1}(U) \subseteq B(\mathcal{H})$ (for $U \subseteq \mathbb{C}$ open) is open in $B(\mathcal{H}) \times B(\mathcal{H})$ as desired.

Example 13.11. Define the elementary matrices $E_{i,j}$ on $\mathcal{H} = \ell^2(\mathbb{N})$ by

$$E_{i,j}(x_1, x_2, \dots) = (0, \dots, 0, x_j, 0, \dots),$$

where I put the x_j on the i th spot. In other words, if $e_k = (0, \dots, 0, 1, 0, \dots)$ denotes the canonical orthonormal basis with the 1 at the k th spot, then $E_{i,j}e_k = \delta_{j,k}e_i$. In terms of only numbers, this means $(E_{i,j}e_k)_l = \delta_{j,k}\delta_{i,l}$. Note that the order of i and j in $E_{i,j}$ might not be the convention that you are used to for elementary matrices; the matrix corresponding to $E_{i,j}$ has a single entry 1 at the i th column and the j th row.

We have that $E_{1,n} \xrightarrow{WOT} 0$ in $B(\mathcal{H})$. Indeed, for all $v, w \in \mathcal{H}$, we have that

$$\langle E_{1,n}v, w \rangle = \langle v_n e_1, w \rangle = \bar{v}_n w_1 \rightarrow 0$$

as $n \rightarrow \infty$. This follows from the fact that $\lim_n v_n = 0$ as $\sum_n |v_n|^2 < \infty$.

On the other hand, $E_{1,n}$ does not converge to zero in the operator norm since $\|E_{1,n}\| = 1$. It follows that it does not converge in norm at all. Indeed, if it would norm converge to some $T \in B(\mathcal{H})$, then it would also WOT-converge to T since the weak operator topology is weaker than the operator norm topology.

Here are some more facts about the weak operator topology that are good exercises:

1. The scalar multiplication map $\mathbb{C} \times B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is WOT-continuous.
2. the map $B(\mathcal{H}) \rightarrow B(\mathcal{H})$ given by $T \mapsto T^*$ is WOT-continuous.
3. Given a fixed $T' \in B(\mathcal{H})$, the map $T \mapsto TT'$ given by right multiplication with T' is WOT-continuous.
4. Similarly, left multiplication is WOT-continuous
5. The multiplication map $B(\mathcal{H}) \times B(\mathcal{H}) \rightarrow B(\mathcal{H})$ is *not* WOT-continuous when \mathcal{H} is infinite-dimensional.
6. Let $S: \mathcal{H} \rightarrow \mathcal{H}$ be the Hilbert hotel operator on $\mathcal{H} = \ell^2(\mathbb{N})$ (usually called the shift). It is defined by $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$ on a sequence $(x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N})$. Then $S^n \xrightarrow{WOT} 0$ converges. On the other hand, the sequence S^n does not converge at all in the operator norm.
7. This is similarly the case for $(S^*)^n$, where $S^*(x_1, x_2, \dots) = (x_2, \dots)$ is the adjoint of S .

13.4 Nets

For a general Hilbert space \mathcal{H} , the weak operator topology on $B(\mathcal{H})$ is not first countable. Therefore, it is in general *not* allowed to use sequences to prove things like continuity of a map, or show that a set is closed. However, you can use nets, which are generalizations of sequences. Nets are indexed by directed sets I , which generalize indexing by the natural numbers in two ways:

1. The cardinality of I can be larger than \mathbb{N} .
2. I is not quite totally ordered.

Definition 13.12. A poset (I, \leq) is *directed* if for every $x, y \in I$ there exists a $z \in I$ such that $x \leq z$ and $y \leq z$. A *net* in a topological space X is a map $x_\bullet: I \rightarrow X$. We say that $x \in X$ is a *limit* of the net x_\bullet if for every open neighbourhood U of x , there exists an $i \in I$ such that for all $j \geq i$ we have $x_j \in U$.

When a topology is of a more analytic nature, spelling out this definition often results in a more ‘epsilon delta style’ condition for being a net. For example, a net $(T_i)_{i \in I}$ converges to $T \in B(\mathcal{H})$ in the weak operator topology if and only if for all $v, w \in \mathcal{H}$ the net $\langle v, T_i w \rangle$ converges to $\langle v, T w \rangle$.

Here are some useful basic facts about nets we won’t prove:

1. a map $f: X \rightarrow Y$ is continuous if and only for every net $(x_i)_{i \in I}$ converging to $x \in X$, the net $(f(x_i))_{i \in I}$ converges to $f(x)$.
2. a subset $S \subseteq X$ is closed if and only if for every net $(x_i)_{i \in I}$ in S that converges in X to some $x \in X$, we have that $x \in S$.
3. As with sequences, in non Hausdorff spaces nets can converge to multiple points. In fact, X is Hausdorff if and only if every convergent net converges only to a single point.

13.5 The ultraweak topology

Let $\mathcal{H}^{\oplus\mathbb{N}} = \bigoplus_{n=0}^{\infty} \mathcal{H}$ be the Hilbert direct sum of countably many copies of \mathcal{H} . Recall that $\mathcal{H}^{\oplus\mathbb{N}}$ consists of \mathcal{H} -valued ℓ^2 -sums, i.e. $(v_n)_{n \in \mathbb{N}}$ in \mathcal{H} with

$$\sum_{n=0}^{\infty} \|v_n\|^2 < \infty,$$

where $\|\cdot\|$ is the norm induced by the inner product of \mathcal{H} . Consider the infinite block diagonal map

$$\Delta: B(\mathcal{H}) \rightarrow B(\mathcal{H}^{\oplus\mathbb{N}}) \quad T \mapsto \begin{pmatrix} T & 0 & \dots \\ 0 & T & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

In other words, $T \in B(\mathcal{H})$ gets mapped to the operator $(v_n)_{n \in \mathbb{N}} \mapsto (Tv_n)_n$. Note that $\Delta(T)$ is linear and defines a bounded operator on $\mathcal{H}^{\oplus\mathbb{N}}$ since

$$\sum_{n=0}^{\infty} \|Tv_n\|^2 \leq \sum_{n=0}^{\infty} \|T\|^2 \|v_n\|^2 = \|T\|^2 \|(v_n)_{n \in \mathbb{N}}\|^2,$$

which is in particular finite. Since $\Delta(1) = 1$, it follows that Δ is bounded of norm 1.

It is clear that Δ is an algebra homomorphism. It is also not difficult to verify that Δ is a $*$ -homomorphism, i.e. that

$$\begin{pmatrix} T & 0 & \dots \\ 0 & T & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}^* = \begin{pmatrix} T^* & 0 & \dots \\ 0 & T^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

It is also obvious that Δ is injective, so we can see $B(\mathcal{H})$ as a subspace of $B(\mathcal{H}^{\oplus\mathbb{N}})$.

Definition 13.13. The *ultraweak (UW) topology* (sometimes called the *σ -weak topology*) is the subspace topology on $B(\mathcal{H})$ for the WOT on $B(\mathcal{H}^{\oplus\mathbb{N}})$.

In other words, it is the weak topology on $B(\mathcal{H})$ for the maps

$$B(\mathcal{H}) \rightarrow B(\mathcal{H}^{\oplus\mathbb{N}}) \xrightarrow{f_{v_n, w_n}} \mathbb{C}$$

where $v_n, w_n \in \mathcal{H}^{\oplus\mathbb{N}}$. Spelling this out, we see that the UW topology is the weakest topology making the maps

$$T \mapsto f_{v_n, w_n}(T) = \langle v_n, Tw_n \rangle_{\mathcal{H}^{\oplus\mathbb{N}}} = \sum_{n=0}^{\infty} \langle v_n, Tw_n \rangle_{\mathcal{H}}$$

continuous for all sequences $v_n, w_n \in \mathcal{H}$ such that

$$\sum_{n=0}^{\infty} \|v_n\|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|w_n\|^2 < \infty.$$

Wait how does this compare with this trace class condition?!

Remark 13.14. It is one of the many unfortunate terminologies in the field of operator topologies on $B(\mathcal{H})$, but the ultraweak topology is *stronger* than the weak operator topology. In fact, it follows that the maps $f_{v,w}: B(\mathcal{H}) \rightarrow \mathbb{C}$ for $v, w \in \mathcal{H}$ are continuous in the UW topology by taking v_n to be the sequence with only a single nonzero entry v and similar for w . It turns out that if \mathcal{H} is infinite-dimensional, then the ultraweak topology is strictly stronger than the WOT.

Remark 13.15. Unlike the basic open sets in the weak operator topology, the basic opens

$$\{T \in B(\mathcal{H}) : \sum_{n=0}^{\infty} \langle v_n, Tw_n \rangle < \epsilon\}$$

of the ultraweak topology do form a neighborhood basis of $0 \in B(\mathcal{H})$.

14 The double commutant theorem (October 23)

The goal of this lecture is to prove the double commutant theorem, which shows that a $*$ -subalgebra $A \subseteq B(\mathcal{H})$ is a von Neumann subalgebra if and only if it is closed in the topologies we have just learned about. We start by proving one direction, which is easier:

Proposition 14.1. *If $A \subseteq B(\mathcal{H})$ is a von Neumann subalgebra, then it is closed in the weak operator topology.*

Proof. Let $T \in \overline{A}^{\text{WOT}}$, where $\overline{A}^{\text{WOT}}$ denotes the WOT-closure in $B(\mathcal{H})$. We need to show that $T \in A = A''$, i.e. that $TT' = T'T$ for all $T' \in A'$. Because T is in the closure of A , every open neighborhood of T intersects A . On basic opens this means that for all $v', w' \in \mathcal{H}$ and for all $\epsilon > 0$, we have that

$$\{T_1 \in B(\mathcal{H}) : |\langle v', (T - T_1)(w') \rangle| < \epsilon\} \cap A \neq \emptyset. \quad (6)$$

It suffices to show that

$$\langle v, TT'w \rangle = \langle v, T'Tw \rangle$$

for all $v, w \in \mathcal{H}$. Let $T' \in A'$. Given any $T_1 \in A$, we can rewrite

$$\begin{aligned} \langle v, TT'w \rangle &= \langle v, (T - T_1)(T'w) \rangle + \langle v, T_1T'w \rangle \\ &= \langle v, (T - T_1)(T'w) \rangle + \langle v, T'T_1w \rangle \\ &= \langle v, (T - T_1)(T'w) \rangle - \langle v, T'(T - T_1)(w) \rangle + \langle v, T'Tw \rangle \\ &= \langle v, (T - T_1)(T'w) \rangle - \langle (T')^*(v), (T - T_1)(w) \rangle + \langle v, T'Tw \rangle, \end{aligned}$$

where in the second equality we used that $T_1 \in A$ commutes with $T' \in A'$. Given any $\epsilon > 0$, we can use (6) for appropriate v' and w' to pick T_1 so that both the first and the second term in the final expression have absolute value smaller than $\epsilon/2$. It follows that

$$|\langle v, TT'w \rangle - \langle v, T'Tw \rangle| < \epsilon$$

for all $v, w \in \mathcal{H}$ and hence $TT' = T'T$ as desired. \square

Since the weak operator topology is weaker than the operator norm, it follows in particular that every von Neumann subalgebra of $B(\mathcal{H})$ is norm closed. Hence every von Neumann subalgebra is a C^* -algebra.

The following result will be useful to prove the double commutant theorem.

Lemma 14.2. *Let $A \subseteq B(\mathcal{H})$ be a $*$ -subalgebra. Then $\Delta(A'') = \Delta(A')$.*

Proof. First note that $\Delta(A)'$ consists of those $T \in B(\mathcal{H}^{\oplus \mathbb{N}})$ with all its entries in A' . More precisely, note that given $T \in B(\mathcal{H}^{\oplus \mathbb{N}})$, one can define the i, j th entry $T_{i,j}$ which is a bounded linear operator on \mathcal{H} . The map $T \mapsto T_{i,j}$ is a bounded linear operator $B(\mathcal{H}^{\oplus \mathbb{N}}) \rightarrow B(\mathcal{H})$ (but not an algebra map). Showing that $T \in \Delta(A)'$ if and only if $T_{i,j} \in A'$ for all i and j follows by a computation similar in spirit to Example 13.7.

The operator $T \in B(\mathcal{H}^{\oplus \mathbb{N}})$ commutes with all $T' \in \Delta(A)'$ if and only if T is block diagonal with entries in A'' . This follows by computing TT' and $T'T$ for the case where $T' = a'E_{ij}$, where $a' \in A'$ and $E_{ij} \in B(\mathcal{H}^{\oplus \mathbb{N}})$ is the operator with $(E_{ij})_{ij} = \text{id}_{\mathcal{H}}$ and $(E_{ij})_{kl} = 0$ for all other entries. \square

Theorem 14.3 (double commutant theorem). *Let $A \subseteq B(\mathcal{H})$ be a $*$ -subalgebra. The following are equivalent:*

1. *A is a von Neumann subalgebra*
2. *A is closed in the weak operator topology*
3. *A is closed in the ultraweak topology*

Proof. We have already shown that $1 \implies 2$ and $2 \implies 3$ follows from the ultraweak topology being stronger than the WOT. It thus remains to show that $3 \implies 1$.

Let $A \subseteq B(\mathcal{H})$ be a $*$ -subalgebra. It suffices to show that $A \subseteq A''$ is dense in the ultraweak topology. Indeed, if A is ultraweakly closed in $B(\mathcal{H})$, we then get that $A = \overline{A}^{\text{UWT}} = A''$ as desired. By definition of the ultraweak topology, this means that we need to show that $\Delta(A)$ is weak operator dense in $\Delta(A'') = \Delta(A)''$. By replacing \mathcal{H} with $\mathcal{H}^{\oplus \mathbb{N}}$ and A by $\Delta(A)$, it thus suffices to show that $2 \implies 1$.

So we want to show that if $A \subseteq B(\mathcal{H})$ is a $*$ -subalgebra, then $A \subseteq A''$ is dense in the weak operator topology. Let $T_0 \in A''$. We need to show that for every $v, w \in \mathcal{H}$ and every $\epsilon > 0$, A intersects the set

$$\{T \in B(\mathcal{H}) : |\langle (T - T_0)(v), w \rangle| < \epsilon\}$$

nontrivially. Since

$$|\langle (T - T_0)(v), w \rangle| \leq \|(T - T_0)(v)\| \|w\|$$

it suffices¹⁴ to show that for every $v \in \mathcal{H}$ and every $\epsilon > 0$ the set

$$\{T \in B(\mathcal{H}) : \|(T - T_0)(v)\| < \epsilon\} \cap A$$

is nonempty. Indeed, we can assume $w \neq 0$ without loss of generality and replace ϵ by $\epsilon/\|w\|$. Let $v \in \mathcal{H}$ and define the subset of \mathcal{H} given by

$$C := \overline{\{a(v) : a \in A\}},$$

where the closure is taken in the topology induced by the inner product on \mathcal{H} . We are done if we can show that $T_0 v \in C$.

Note that $C \subseteq \mathcal{H}$ is a closed subspace and $a \cdot C \subseteq C$ for all $a \in A$. Indeed, if $\lim_{n \rightarrow \infty} a_n v = w$ for some $a_n \in A$, then $\lim_{n \rightarrow \infty} a a_n v = aw$ by continuity of the operator $a : \mathcal{H} \rightarrow \mathcal{H}$. Moreover, if the sequence $b_n v$ additionally converges to u , then $(a_n + b_n)(v)$ converges to $w + u$.

It now follows by the next lemma that the projection operator $p \in B(\mathcal{H})$ onto C lies in A' . We can now finish the proof by the computation

$$pT_0(v) = T_0 p(v) = T_0(v).$$

The first equality follows from $T_0 \in A''$ and $p \in A'$. The second equality follows from the fact that $v \in C$ since we can take $a = 1 \in A$. We have proven that $T_0(v)$ is in the image of p and hence $T_0(v) \in C$. \square

Lemma 14.4. *Let $A \subseteq B(\mathcal{H})$ be a $*$ -subalgebra. Let $C \subseteq \mathcal{H}$ be a closed subspace such that $aC \subseteq C$ for all $a \in A$. Then the projection p onto C lies in A' .*

¹⁴Remark for people who know what this means: we are showing that A is dense in the strong operator topology, which is stronger than the weak operator topology. The strong operator topology is convenient in the current context as it is the topology of pointwise convergence (of nets!) only using the topology of \mathcal{H} .

Proof. Let $a \in A$ and $v \in \mathcal{H}$. We need to verify that $ap(v) = pa(v)$. Recall that there is an orthogonal decomposition $\mathcal{H} \cong C \oplus C^\perp$ and so we can write $v = v_1 + v_2$ with $v_1 \in C$ and $v_2 \in C^\perp$. We have that $p(v_1) = v_1$, while $p(v_2) = 0$. It suffices to check the equality $ap(v) = pa(v)$ separately for the case where $v = v_1 \in C$ and $v = v_2 \in C^\perp$. In the former case we have

$$ap(v) = a(v) = pa(v)$$

as also $a(v) \in C$ by assumption.

The latter case now follows by the same proof using the fact that $C^\perp \subseteq \mathcal{H}$ is also a closed subspace such that $aC^\perp \subseteq C^\perp$. Indeed, if $v \in C^\perp$, then

$$\langle av, w \rangle = \langle v, a^*w \rangle = 0$$

for all $w \in C$ because $a^* \in A$. Hence $av \in C^\perp$ and we are done. □

15 Trace class operators (October 28)

Our goal is to define the trace of a linear map $T: \mathcal{H} \rightarrow \mathcal{H}$. If \mathcal{H} is infinite dimensional, this trace should not be finite for a general bounded operator T . For example on $\ell^2(\mathbb{N})$, we should certainly have

$$\mathrm{Tr}(\mathrm{id}_{\mathcal{H}}) = \mathrm{Tr} \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = 1 + 1 + \dots = \infty.$$

But it can be even worse. For example, for any bounded sequence λ_n we can define a diagonal bounded operator on $\ell^2(\mathbb{N})$ and its trace should be $\sum_{n=0}^{\infty} \lambda_n$. But this sum doesn't need to have any convergence properties, e.g. for $\lambda_n = (-1)^n$, we get

$$\begin{pmatrix} 1 & 0 & \dots \\ 0 & -1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = 1 - 1 + \dots = ?$$

In this lecture, we will identify a subset $TC(\mathcal{H}) \subseteq B(\mathcal{H})$ consisting of ‘operators of finite trace’. Such operators are called *trace class*. More specifically, we want trace class operators to generalize the case of absolutely convergent series from diagonal operators to arbitrary operators. We choose absolutely convergent series because they are much better behaved than arbitrary convergent series. Moreover, we are able to make absolutely convergent series into a Banach space using the sum of the absolute values

There are several approaches to trace class operators. The most common and perhaps most intuitive approach is to let $\{e_i\}_{i \in I}$ be an orthonormal basis of \mathcal{H} and attempt to define

$$\mathrm{Tr} T = \sum_{i \in I} \langle e_i, T e_i \rangle. \tag{7}$$

Since we want to generalize absolutely convergent series, we want to require that $\mathrm{Tr} |T| < \infty$. Here we arrive at the problem that we don't know what the absolute value of an operator is. This is not a big deal however as it can be defined using continuous functional calculus as we will see in a few lectures. We do have to show that the trace is independent of the choice of basis and that $\mathrm{Tr} |T| < \infty$ implies $\mathrm{Tr} T < \infty$, which will require more functional analytic technology.

We will instead follow an alternate route which leads us quicker to the desired results about preduals we will need to understand abstract von Neumann algebras in the next lecture. The main idea is to start out with finite rank operators—which certainly have finite trace—and see for what kind of infinite linear combinations of them the trace still makes sense.

We say that a bounded linear operator $T: \mathcal{H} \rightarrow \mathcal{H}$ has rank 1 if its image is one-dimensional. For $v, w \in \mathcal{H}$ define the operator

$$E_{v,w}(u) = v \langle w, u \rangle.$$

You can check that these operators generalize the elementary matrices $E_{i,j}$ of Example 13.11 from the case where $v = e_i$ and $w = e_j$ are elements of an orthonormal basis.

The operator $E_{v,w}$ is bounded of norm $\|v\| \|w\|$ since

$$\|E_{v,w}(u)\| \leq \|v\| \|w\| \|u\|$$

by Cauchy-Schwarz and

$$\|E_{v,w}(w)\| = \|v\| \|w\| \|w\|.$$

Note that $E_{v,w}$ has rank at most 1. It follows from the above norm computation that $E_{v,w}$ has rank 1 exactly when v and w are both nonzero.

In fact, all rank 1 operators are of the form $E_{v,w}$. To see this, recall that it follows from the Riesz-Fréchet theorem that every bounded linear functional $f: \mathcal{H} \rightarrow \mathbb{C}$ is of the form $f(u) = \langle w, u \rangle$ for some $u \in \mathcal{H}$. Now the claim follows by realizing that T has rank at most 1 if and only if it factors through a linear functional.

More generally, the vector space of finite rank operators is the (nonclosed) span of the operators $E_{v,w}$. We want to take an ℓ^1 -sum completion of this vector space. In other words, we want to define a norm on this space *different*¹⁵ from the operator norm and complete under that.

Formally, we do this as follows. Let X be the (huge) vector space of formal expressions

$$\sum_{v,w \in \mathcal{H} \setminus \{0\}} \lambda_{v,w} e_{v,w}$$

where $\lambda_{v,w} \in \mathbb{C}$ is a collection of numbers such that

$$\sum_{v,w \in \mathcal{H} \setminus \{0\}} |\lambda_{v,w}| \|v\| \|w\| < \infty$$

with the norm

$$\left\| \sum_{v,w \in \mathcal{H} \setminus \{0\}} \lambda_{v,w} e_{v,w} \right\| := \sum_{v,w \in \mathcal{H} \setminus \{0\}} |\lambda_{v,w}| \|v\| \|w\|.$$

It is not difficult to show this defines a complete norm on X .

The subspace Y of sums for which only finitely many $\lambda_{v,w}$ are nonzero is dense in X . Consider the unique linear operator $\tilde{\mu}: Y \rightarrow B(\mathcal{H})$ given by $e_{v,w} \mapsto E_{v,w}$ on the canonical basis. Its image is given by the finite rank operators on \mathcal{H} . By construction, we have $\|\tilde{\mu}(e_{v,w})\| = \|v\| \|w\| = \|e_{v,w}\|$ and so $\tilde{\mu}$ is bounded of norm 1. By Lemma 4.9, $\tilde{\mu}$ extends uniquely to a norm 1 operator $\mu: X \rightarrow B(\mathcal{H})$.

Definition 15.1. An operator $T \in B(\mathcal{H})$ is called *trace class* if it is in the image of μ . Let $TC(\mathcal{H}) = \mu(X)$ be the vector space of trace class operators.

Note that the kernel of μ is huge. For starters, we have the relations

$$E_{v_1+v_2,w} = E_{v_1,w} + E_{v_2,w} \quad E_{v,w_1+w_2} = E_{v,w_1} + E_{v,w_2} \quad E_{\lambda v,w} = \lambda E_{v,w} \quad E_{v,\lambda w} = \bar{\lambda} E_{v,w}, \quad (8)$$

so that we have $e_{v_1+v_2,w} - e_{v_1,w} - e_{v_2,w} \in \ker \tilde{\mu} \subseteq \ker \mu$ and similar for the other expressions.

We want to give $TC(\mathcal{H})$ a norm. By the first isomorphism theorem for vector spaces, we have

$$\frac{X}{\ker \mu} \cong TC(\mathcal{H}).$$

Since $\ker \mu \subseteq X$ is a closed subspace, we can therefore give trace class operators the corresponding quotient norm. Explicitly, the *trace norm* is given by

$$\|T\|_1 := \inf \left\{ \sum_{v,w \in \mathcal{H}} \|v\| \|w\| : T = \sum_{v,w \in \mathcal{H}} E_{v,w} \text{ with } \sum_{v,w \in \mathcal{H}} \|v\| \|w\| < \infty \right\},$$

¹⁵We will see that this norm is also different from the subspace topology on trace class operators obtained from the operator topologies on $B(\mathcal{H})$ we have defined in previous lectures. In fact, the norm we define will induce a strictly stronger topology than the operator norm, which is therefore also strictly stronger than both the weak operator topology and the ultraweak topology restricted to $TC(\mathcal{H})$.

where we could leave out the $\lambda_{v,w}$ factor in front of the $E_{v,w}$ as we can absorb it as $\lambda_{v,w}E_{v,w} = E_{\lambda_{v,w}v,w}$.

Note that $T \in TC(\mathcal{H})$ if and only if the set in the infimum is not empty and so $\|T\|$ is finite in that case. On the other hand, there can be many ways to write T in the desired form, e.g. for $T = 0$, we could write $T = E_{v,w} + E_{-v,w}$ and $\|e_{v,w} + e_{-v,w}\| = \|v\|\|w\| + \|v\|\|w\|$ can be nonzero even though $\|T\|_1 = 0$.

One can show that for any element

$$\sum_{v,w \in \mathcal{H} \setminus \{0\}} \lambda_{v,w} e_{v,w}$$

of X only countably many $\lambda_{v,w}$ can be nonzero. It follows that any trace class operator can be written as a countable sum

$$T = \sum_{n \in \mathbb{N}} E_{v_n, w_n}$$

for some sequences $v_n, w_n \in \mathcal{H}$ such that

$$\sum_{n \in \mathbb{N}} \|v_n\|\|w_n\| < \infty.$$

Note that even though the norm of $TC(\mathcal{H})$ agrees with the norm of $B(\mathcal{H})$ on rank one operators, the inclusion $TC(\mathcal{H}) \hookrightarrow B(\mathcal{H})$ is still not an isometry. For example, if e_1 and e_2 are orthonormal vectors in \mathcal{H} , we have

$$\|E_{e_1, e_1} + E_{e_2, e_2}\|_1 = 2 \neq 1 = \|E_{e_1, e_1} + E_{e_2, e_2}\|,$$

where on the right we used the operator norm. More generally, for diagonal operators, the trace norm is the ℓ^1 -norm, while the operator norm is the ℓ^∞ -norm.

It also follows from these considerations that the subspace $TC(\mathcal{H})$ of $B(\mathcal{H})$ is not closed.¹⁶ Explicitly, take λ_n to be a bounded sequence and let T_{λ_n} be the corresponding diagonal operator on $\mathcal{H} = \ell^2(\mathbb{N})$. Let $\lambda^{(k)} = (\lambda_1, \dots, \lambda_k, 0, \dots)$ be the sequence λ_n except that everything becomes zero after the k th entry. Then the operator $T_{\lambda^{(k)}}$ is of finite rank for all k . Note that in general $T_{\lambda^{(k)}}$ does not converge to T_{λ_n} as $k \rightarrow \infty$ in the operator norm. For example, if we take $\lambda_n = 1$ for all n we get $T_{\lambda_n} = \text{id}_{\mathcal{H}}$ but $T_{\lambda_n} - T_{\lambda^{(k)}}$ has norm 1 for all k . In fact, the identity on \mathcal{H} cannot be written as the limit of finite rank operators at all. However, if $\lim_{n \rightarrow \infty} \lambda_n = 0$, then $T_{\lambda^{(k)}} \rightarrow T_{\lambda_n}$ as $k \rightarrow \infty$ ¹⁷. But the condition that $\lim_{n \rightarrow \infty} \lambda_n = 0$ is not enough for the diagonal operator to be trace class; for that we need $\sum_n |\lambda_n| < \infty$. So by taking for example $\lambda_n = 1/n$, we see that there exists a sequence $T_{\lambda^{(k)}}$ of finite rank operators which converges in $B(\mathcal{H})$ but does not converge in $TC(\mathcal{H})$.

We now want to define a trace operation $\text{Tr}: TC(\mathcal{H}) \rightarrow \mathbb{C}$. The idea is again to define it on the rank 1 operators and then extend by continuity. For this, we first need to decide what should be the trace of $E_{v,w}$. There are several ways to compute it, for example using (7), which apparently makes sense here:

$$\begin{aligned} \sum_i \langle e_i, E_{v,w}(e_i) \rangle &= \sum_i \langle e_i, v \langle w, e_i \rangle \rangle = \sum_i \langle e_i, v \rangle \langle w, e_i \rangle = \sum_{i,j,k} \langle e_i, v_j e_j \rangle \langle w_k e_k, e_i \rangle \\ &= \sum_{i,j,k} v_j \overline{w_k} \delta_{i,j} \delta_{k,i} = \sum_i v_i \overline{w_i} = \langle w, v \rangle. \end{aligned}$$

¹⁶This might be confusing because $TC(\mathcal{H})$ is a complete subspace of $B(\mathcal{H})$. But we have seen that the restriction of the norm of $B(\mathcal{H})$ to $TC(\mathcal{H})$ is different from the trace class norm, so there is no contradiction.

¹⁷More generally (for those who know what that means) a bounded linear operator T can be written as the limit of finite rank operators if and only if it is compact

So we should define $\text{Tr } E_{v,w} = \langle w, v \rangle$. Note that confusingly we have to exchange v and w in this formula. This is not that surprising once you realize that the trace needs to be complex linear, but $E_{v,w}$ is linear in v and anti-linear in w .

We thus define $\text{Tr}: Y \rightarrow \mathbb{C}$ by

$$\text{Tr} \left(\sum_{v,w} \lambda_{v,w} e_{v,w} \right) = \sum_{v,w} \lambda_{v,w} \langle w, v \rangle.$$

By Cauchy-Schwarz, we have that

$$\left| \text{Tr} \left(\sum_{v,w} \lambda_{v,w} e_{v,w} \right) \right| \leq \sum_{v,w} |\lambda_{v,w}| \|v\| \|w\| = \left\| \sum_{v,w} \lambda_{v,w} e_{v,w} \right\|,$$

so that Tr extends to a bounded operator $X \rightarrow \mathbb{C}$ of norm 1.

To show that this induces a trace on $TC(\mathcal{H}) \cong Y / \ker \mu$, we have to show that $\text{Tr}: X \rightarrow \mathbb{C}$ vanishes on the kernel of μ . It follows by the sesquilinearity of the inner product that Tr vanishes on $\ker \tilde{\mu}$. To show that it vanishes on all of $\ker \mu$ we will use a continuity argument. We postpone this to the next lecture, where we will define a more general trace pairing

$$B(\mathcal{H}) \times TC(\mathcal{H}) \rightarrow \mathbb{C}$$

by $(T, T') \mapsto \text{Tr}(T \circ T')$.

16 Abstract von Neumann algebras and preduals (October 30)

In this lecture we will study an intrinsic definition of von Neumann algebra which does not depend on representing it as operators on a Hilbert space.

Definition 16.1. A(n abstract) *von Neumann algebra* is a C^* -algebra A with the property that it has a *predual*: a Banach space L^1A such that $(L^1A)^\vee \cong A$ isometrically as Banach spaces. Here we equip $(L^1A)^\vee$ with the usual operator norm topology on functionals.

Remark 16.2. It turns out that the predual of a C^* -algebra is unique, see [5, Corollary 1.13.3.]. This is not immediately obvious. In fact, general Banach spaces can have multiple preduals (or none at all).

The goal of the rest of the lecture is to show that the von Neumann subalgebras of $B(\mathcal{H})$ we studied in the previous lectures are von Neumann algebras in the above sense. More specifically, we sketch the proof of the following result.

Theorem 16.3. *Let $A \subseteq B(\mathcal{H})$ be a von Neumann subalgebra. Then*

$$L^1A = \{ UW \text{ continuous linear functionals } A \rightarrow \mathbb{C} \}$$

with the operator norm topology on A^\vee is a predual. The isomorphism $ev: A \rightarrow (L^1A)^\vee$ is given by the evaluation map $ev_a(f) = f(a)$. Moreover, under this isomorphism the ultraweak topology on A corresponds to the weak topology on $(L^1A)^\vee$.*

Note that we could talk about a map $A \rightarrow \mathbb{C}$ as being ultraweakly continuous, as we can use the subspace topology on A from $B(\mathcal{H})$ with the ultraweak topology. Moreover, since the ultraweak topology is weaker than the norm topology, these ultraweakly continuous functionals are contained in A^\vee , which itself has an operator norm.

Before turning to general A , we will use trace class operators to show $B(\mathcal{H})$ itself is a von Neumann algebra:

Theorem 16.4. *The Banach space $TC(\mathcal{H})$ is a predual of $B(\mathcal{H})$. The isomorphism $B(\mathcal{H}) \rightarrow TC(\mathcal{H})^\vee$ sends $T \in B(\mathcal{H})$ to the trace pairing $T' \mapsto \text{Tr}(T'T)$ where $T' \in TC(\mathcal{H})$. Moreover, the ultraweak topology on $B(\mathcal{H})$ corresponds to the weak*-topology on $TC(\mathcal{H})^\vee$.*

For the trace pairing to make sense, we need to be able to take the trace of the composition $T'T$, where $T' \in TC(\mathcal{H})$ and $T \in B(\mathcal{H})$:

Proposition 16.5. *$TC(\mathcal{H})$ is a two-sided *-ideal in $B(\mathcal{H})$.*

Proof. Let $T \in B(\mathcal{H})$ and $T' \in TC(\mathcal{H})$. Pick $v_n, w_n \in \mathcal{H}$ sequences of vectors so that

$$\sum_{n \in \mathbb{N}} \|v_n\| \|w_n\| < \infty$$

and

$$T' = \sum_{n \in \mathbb{N}} E_{v_n, w_n}.$$

Indeed, recall that we can always take this sum to be countable. For all $u \in \mathcal{H}$ we have that

$$T \left(\sum_n v_n \langle w_n, u \rangle \right) = \sum_n T(v_n) \langle w_n, u \rangle.$$

This follows for finite sums by linearity of T . It follows for arbitrary sums by continuity of T , since the series $\sum_n v_n \langle w_n, u \rangle$ converges in \mathcal{H} by Cauchy-Schwarz. We see that $TT' = \sum_{n \in \mathbb{N}} E_{Tv_n, w_n} \in TC(\mathcal{H})$. Analogously, we find that $T'T = \sum_{n \in \mathbb{N}} E_{v_n, T^*w_n} \in TC(\mathcal{H})$ using

$$\sum_n v_n \langle w_n, T(u) \rangle = \sum_n v_n \langle T^*(w_n), u \rangle.$$

Since $TC(\mathcal{H}) \subseteq B(\mathcal{H})$ is a linear subspace, we have shown that it is an ideal.

To show that $TC(\mathcal{H})$ is a $*$ -ideal, it suffices to show that

$$(T')^* = \sum_n E_{w_n, v_n}.$$

For this, first note that $E_{v_n, w_n}^* = E_{w_n, v_n}$:

$$\langle w \langle v, u_1 \rangle, u_2 \rangle = \overline{\langle v, u_1 \rangle} \langle w, u_2 \rangle = \langle u_1, v \rangle \langle w, u_2 \rangle = \langle u_1, v \langle w, u_2 \rangle \rangle.$$

By linearity, we obtain the desired equality for finite sums. By continuity of the star operation and the fact that finite rank operators are dense in $TC(\mathcal{H})$ for the operator norm, our claim follows for all trace class operators. \square

Next we will show that the trace is well defined, i.e. that it vanishes on the kernel of μ . For this we will need a bunch of lemmas.

Lemma 16.6. *An element $T \in B(\mathcal{H})$ is trace class if and only if it can be written as*

$$T = \sum_{n \in \mathbb{N}} E_{v_n, w_n}$$

with v_n, w_n sequences of vectors in \mathcal{H} such that

$$\sum_n \|v_n\|^2 < \infty \text{ and } \sum_n \|w_n\|^2 < \infty.$$

Proof. Suppose v_n, w_n satisfy

$$\sum_n \|v\|^2 < \infty \text{ and } \sum_n \|w\|^2 < \infty.$$

We then have that

$$\sum_n \|v_n\| \|w_n\| \leq \left(\sum_n \|v_n\| \right) \left(\sum_k \|w_k\| \right) \leq \sqrt{\sum_n \|v_n\|^2} \sqrt{\sum_k \|w_k\|^2} < \infty.$$

Therefore $\sum_n E_{v_n, w_n}$ does define a trace class operator.

Conversely, suppose T is a trace class operator. Then

$$T = \sum_n E_{v_n, w_n}$$

for some v_n, w_n satisfying

$$\sum_n \|v_n\| \|w_n\| < \infty.$$

Using the bilinearity property of rank 1 operators, we can rewrite

$$T = \sum_n E_{\sqrt{\frac{\|w_n\|}{\|v_n\|}} v_n, \sqrt{\frac{\|v_n\|}{\|w_n\|}} w_n},$$

where

$$\sum_n \left(\sqrt{\frac{\|w_n\|}{\|v_n\|}} \|v_n\| \right)^2 = \sum_n \|v_n\| \|w_n\| = \sum_n \left(\sqrt{\frac{\|v_n\|}{\|w_n\|}} \|w_n\| \right)^2 < \infty.$$

□

As remarked in the previous lecture, finite rank operators are not norm dense in bounded operators. They are ultraweakly dense however:

Lemma 16.7. *Finite rank operators are ultraweakly dense in $B(\mathcal{H})$.*

Proof. Let $T \in B(\mathcal{H})$. We want to find a net of finite rank operators which converges to T .

Let $\{e_\alpha : \alpha \in A\}$ be an orthonormal basis of \mathcal{H} . If $A_0 \subseteq A$ is a finite subset, then

$$p_{A_0} := \sum_{\alpha \in A_0} E_{e_\alpha, e_\alpha}$$

is the projection onto the finite-dimensional subspace $\text{span}\{e_\alpha : \alpha \in A_0\}$. Since p_{A_0} is finite rank, it is trace class. Using formulas proved in Proposition 16.5 we can therefore write

$$T p_{A_0} = \sum_{\alpha \in A_0} E_{T e_\alpha, e_\alpha},$$

which is also finite rank. Let I be the collection of finite subsets $A_0 \subseteq A$. Then I is directed when ordered under inclusion. We will show that the net $(T p_{A_0})_{A_0 \in I}$ converges to T ultraweakly.

Let $v_n, w_n \in \mathcal{H}$ be sequences of vectors such that

$$\sum_n \|v_n\|^2 < \infty \quad \text{and} \quad \sum_n \|w_n\|^2 < \infty.$$

We need to show that

$$\sum_{n \in \mathbb{N}} |\langle v_n, T(w_n) - T p_{A_0}(w_n) \rangle| \leq \sum_n \|v_n\| \|T\| \|(1 - p_{A_0})(w_n)\|$$

goes to zero as A_0 gets large. We have

$$\sum_n \|w_n\|^2 = \sum_{n \in \mathbb{N}, \alpha \in A} \|w_n^\alpha\|^2 \tag{9}$$

when writing

$$w_n = \sum_{\alpha \in A} w_n^\alpha e_\alpha$$

for all n in terms of the orthonormal basis. Note that $1 - p_{A_0}$ is the projection onto the complement of the span of the e_α with $\alpha \in A_0$. Therefore our task is to choose A_0 so large that

$$1 - p_{A_0}(w_n) = \sum_{\alpha \notin A_0} w_n^\alpha e_\alpha$$

becomes very small.

Since we have better control over sums with $\|w_n\|^2$ in it instead of $\|w_n\|$, we use the same trick as in the proof of Lemma 16.6 to get

$$\sum_n \|v_n\| \|T\| \|(1 - p_{A_0})(w_n)\| \leq \|T\| \sqrt{\sum_n \|v_n\|^2} \sqrt{\sum_n (1 - p_{A_0})\|w_n\|^2}.$$

It therefore suffices to show that the net $(\sum_n (1 - p_{A_0})\|w_n\|^2)_{A_0 \in I}$ converges to zero as $A_0 \rightarrow \infty$.

Let $\epsilon > 0$. Using the fact that the sum \mathfrak{g} is finite, we can pick a finite set $F \subseteq A \times \mathbb{N}$ with the property that for every finite set $F' \subseteq A \times \mathbb{N}$ that contains F we have

$$\sum_{(n,\alpha) \notin F'} \|w_n^\alpha\|^2 < \epsilon.$$

Let $A_0 \subseteq A$ be the (finite) subset of $\alpha \in A$ such that $(\alpha, n) \in F$ for some $n \in \mathbb{N}$. We then have

$$\sum_n (1 - p_{A_0})\|w_n\|^2 = \sum_n \sum_{\alpha \notin A_0} \|w_n^\alpha\|^2 \leq \sum_{\alpha \notin A_0} \|w_n^\alpha\|^2 \leq \sum_{(\alpha,n) \notin F} \|w_n^\alpha\|^2 < \epsilon$$

for any finite subset $A'_0 \subseteq A$ which contains A_0 . □

We are now ready to show the trace is well defined

Proposition 16.8. *Let $T \in B(\mathcal{H})$. The expression*

$$\sum_{v,w} \lambda_{v,w} e_{v,w} \mapsto \sum_{v,w} \lambda_{v,w} \langle w, Tv \rangle$$

is independent of how one presents the trace class operator $T' = \sum_{v,w} E_{v,w}$ as a sum of rank one matrices.

Proof. We need to show that the bounded linear functional $X \rightarrow \mathbb{C}$ defined by

$$\sum_{v,w} \lambda_{v,w} e_{v,w} \mapsto \sum_{v,w} \lambda_{v,w} \langle w, Tv \rangle$$

vanishes on $\ker \mu$ for all $T \in B(\mathcal{H})$. Let $x = \sum_{v,w} \lambda_{v,w} e_{v,w} \in \ker \mu$. We can assume $\lambda_{v,w} = 1$ using bilinearity of $E_{v,w}$. By Lemma 16.6 we can also assume without loss of generality that

$$\sum_n \|v\|^2 < \infty \quad \text{and} \quad \sum_n \|w\|^2 < \infty.$$

In that way, we obtain that the map $B(\mathcal{H}) \rightarrow \mathbb{C}$ given by

$$T \mapsto \sum_{v,w} \langle w, Tv \rangle$$

is ultraweakly continuous. Using Lemma 16.7, it it therefore suffices to check that $\sum_{v,w} \langle w, Tv \rangle$ vanishes when T is finite rank. Since the expression is linear in T , we therefore are only left to check the case $T = E_{v',w'}$. We can do this by hand:

$$\begin{aligned} \sum_n E_{v_n, w_n} = 0 &\implies \sum_n E_{v_n, w_n}(v') = \sum_n v_n \langle w_n, v' \rangle = 0 \\ &\implies \sum_n \langle w_n, E_{v', w'}(v_n) \rangle = \sum_n \langle w_n, v' \langle w', v_n \rangle \rangle = \sum_n \langle w_n, v' \rangle \langle w', v_n \rangle = \sum_n \langle w', v_n \langle w_n, v_n \rangle \rangle = 0 \end{aligned}$$

In the last equation, we used that we can pull the infinite sum into the inner product, using continuity of the map $\langle w', - \rangle: \mathcal{H} \rightarrow \mathbb{C}$. We have thus shown that $\sum_{v,w} \lambda_{v,w} \langle w, Tv \rangle = 0$ whenever $T \in B(\mathcal{H})$ and $\sum_{v,w} \lambda_{v,w} e_{v,w} \in \ker \mu$ as desired. \square

The trace satisfies the usual tracial property:

Lemma 16.9. *For every $T \in B(\mathcal{H})$ and $T' \in TC(\mathcal{H})$, we have $\text{Tr}(T'T) = \text{Tr}(TT')$.*

Proof. From the computations in the previous proof, we have that

$$\text{Tr}(T'T) = \sum_i \langle T^* w_i, v_i \rangle \stackrel{?}{=} \sum_i \langle w_i, T v_i \rangle = \text{Tr}(TT').$$

These are equal if $T' = \sum_i E_{v_i, w_i}$ is finite rank and the general case follows by taking limits on both sides. \square

Proof of Theorem 16.4. Let $T \in B(\mathcal{H})$. The functional $T' \mapsto \text{Tr}(TT')$ on $TC(\mathcal{H})$ is clearly linear and well-defined by Proposition 16.8. To show that $\text{Tr}(T \circ (-)) \in TC(\mathcal{H})^\vee$, we still have to show it is continuous. We thus compute

$$\|\text{Tr}(T \circ (-))\| = \sup_{\substack{v_i, w_i: \\ \sum \|v_i\| \|w_i\| = 1}} \left| \sum_i \langle w_i, T v_i \rangle \right| \leq \sup_{\substack{v_i, w_i: \\ \sum \|v_i\| \|w_i\| = 1}} \sum_i \|w_i\| \|T\| \|v_i\| = \|T\|,$$

so this functional is indeed bounded of norm at most $\|T\|$. Since we eventually also need to show that $T \mapsto \text{Tr}(T \circ (-))$ is norm preserving, we will show its norm equals $\|T\|$. We do this by restricting the supremum to sums with one entry and the case $w_i = T v_i$:

$$\|\text{Tr}(T \circ (-))\| \geq \sup_{v \neq 0} \frac{|\langle T v, T v \rangle|}{\|v\| \|T v\|} = \sup_{v \neq 0} \frac{\|T v\|}{\|v\|} = \|T\|$$

Since the assignment $T \mapsto \text{Tr}(T \circ (-))$ is linear, we have thus shown that there is an isometry $B(\mathcal{H}) \rightarrow TC(\mathcal{H})^\vee$. As all isometries are injective, we are done once we can show surjectivity.

Let $P: TC(\mathcal{H}) \rightarrow \mathbb{C}$ be a continuous linear functional. We want to find $T \in B(\mathcal{H})$ such that $P(\sum_i E_{v_i, w_i}) = \sum_i \langle w_i, T v_i \rangle$ for all sequences v_i, w_i such that

$$\sum_i \|v_i\| \|w_i\| < \infty.$$

Let $v \in \mathcal{H}$. Then $w \mapsto P(E_{v,w})$ defines a \mathbb{C} -antilinear functional $\mathcal{H} \rightarrow \mathbb{C}$. It is bounded since

$$\|P(E_{v,w})\| \leq \|P\| \|E_{v,w}\| = \|P\| \|v\| \|w\|.$$

By the Riesz-Fréchet theorem, there exists a unique vector $Tv \in \mathcal{H}$ such that $P(E_{v,w}) = \langle w, Tv \rangle$ for all $w \in \mathcal{H}$.

We claim that $v \mapsto Tv$ is the desired preimage. Note that T is linear by the bilinearity of the inner product. The equation $P(\sum_i E_{v_i, w_i}) = \sum_i \langle w_i, Tv_i \rangle$ hold for finite sums, and therefore for arbitrary sums as finite rank operators are dense in $TC(\mathcal{H})$. Finally, note that $T \in B(\mathcal{H})$ is bounded since

$$\|Tv\|^2 = \langle Tv, Tv \rangle = P(E_{v, Tv}) \leq \|P\| \|v\| \|Tv\| \implies \|Tv\| \leq \|P\| \|v\|.$$

We have shown that the trace pairing defines an isometric isomorphism $B(\mathcal{H}) \rightarrow TC(\mathcal{H})^\vee$.

Finally we need to show that the ultraweak topology on $B(\mathcal{H})$ corresponds to the weak* topology on $TC(\mathcal{H})^\vee$. The weak* topology is the weakest topology making the evaluation functionals $B(\mathcal{H}) \rightarrow \mathbb{C}$ given by $T' \mapsto \text{Tr}(TT')$ continuous for all $T' \in TC(\mathcal{H})$. For a trace class operator of the form

$$T' = \sum_{n \in \mathbb{N}} E_{v_n, w_n}$$

the functional $\text{Tr}(T' \circ (-))$ is given by

$$T \mapsto \sum_n \langle w_n, Tv_n \rangle.$$

These are exactly the functionals we require to be continuous in the ultraweak topology, with the only exception that the condition on the sequences w_n and v_n is slightly different for the ultraweak topology than for being trace class. This difference is resolved in Lemma 16.6. \square

Remark 16.10. For those people who know about tensor products, the fact that the only relations among finite rank operators are $E_{\lambda v_1 + v_2, w} = \lambda E_{v_1, w} + E_{v_2, w}$ and $E_{v, \lambda w_1 + w_2} = \bar{\lambda} E_{v, w_1} + E_{v, w_2}$ is expressing the fact that the finite rank operators on \mathcal{H} are given by the algebraic tensor product $\mathcal{H} \otimes \bar{\mathcal{H}}$. By algebraic tensor product we mean the tensor product of vector spaces; we don't complete the tensor product under a norm. The complex conjugate of a Hilbert space is the same Hilbert space except that its scalar multiplication and inner product complex are conjugated.

In fact, trace class operators are also a sort of tensor product of \mathcal{H} and $\bar{\mathcal{H}}$, but this tensor product is completed. More specifically, it is the projective tensor product of Banach algebras. This tensor product is also different from the tensor product of Hilbert spaces, in case you know what that is (this would give Hilbert-Schmidt operators instead).

An advantage of viewing trace class operators in this way, is that there is an adjunction in the category of Banach spaces and norm decreasing maps

$$\text{Hom}(X \otimes_\pi Y, Z) \cong \text{Hom}(X, B(Y, Z))$$

for all Banach spaces X, Y, Z . In other words: bounded operators form an internal hom in this category of Banach spaces. It follows that there is also an isometry

$$B(X \otimes_\pi Y, Z) \cong B(X, B(Y, Z)) \tag{10}$$

of Banach spaces. If you already know these facts, the proof of Theorem 16.4 is rather short:

$$TC(\mathcal{H})^\vee = B(\mathcal{H} \otimes_\pi \overline{\mathcal{H}}, \mathbb{C}) \cong B(\mathcal{H}, B(\overline{\mathcal{H}}, \mathbb{C})) \cong B(\mathcal{H}, \mathcal{H}).$$

Here the last isomorphism follows by the Riesz-Fréchet theorem, which says that there is an isometric isomorphism $\mathcal{H} \cong B(\overline{\mathcal{H}}, \mathbb{C})$. However, the proof we gave of Theorem 16.4 essentially generalizes to prove (10), so there is not much reason to first prove (10) except for a conceptual framework.

Example 16.11 (3.2.1 in [2]). We show that the ultraweak topology is strictly stronger than the weak operator topology on $B(\mathcal{H})$ in case $\mathcal{H} = \ell^2(\mathbb{N})$.

Consider the operator

$$T' = \sum_{n \in \mathbb{N}} \frac{1}{n^2} E_{e_n, e_n},$$

where e_n is the standard orthonormal basis of $\ell^2(\mathbb{N})$. Since the sum $\sum_n 1/n^2$ converges, T' is trace class.

Note that given $v \in \mathcal{H}$, we have that

$$\text{Tr}(T' E_{v, v}) = \sum_n \frac{1}{n^2} \langle e_n, E_{v, v}(e_n) \rangle = \sum_n \frac{1}{n^2} \langle e_n, v \bar{v}_n \rangle = \sum_n \frac{1}{n^2} v_n \bar{v}_n$$

is nonzero unless $v = 0$. Recall that the projection p_v onto the one-dimensional subspace spanned by v is given by $E_{v, v}$. Let p be a nonzero projection and let $v \in \mathcal{H} \setminus \{0\}$ be in the image of p . We have that $p \geq p_v$. It will be a consequence of results of the next lecture (see ??) that $\langle w, T w \rangle \geq 0$ for all $w \in \mathcal{H}$ if T is positive. Therefore we obtain that

$$\text{Tr}(T' p) = \sum_n \frac{1}{n^2} \langle e_n, p(e_n) \rangle \geq \sum_n \frac{1}{n^2} \langle e_n, p_v(e_n) \rangle > 0$$

is nonzero for all nonzero projections p .

Let \mathcal{F} be the collection of all projections on finite-dimensional subspaces. This is a directed set when ordered by inclusion of subspaces (or \leq of the corresponding projections). Define a net indexed by $p \in \mathcal{F}$ as

$$T_p = \frac{1 - p}{\text{Tr}(T'(1 - p))}.$$

This net converges to zero in the weak operator topology. Indeed, if $v, w \in \mathcal{H}$, then for all projections $p \geq p_w$, we have

$$|\langle v, T_p w \rangle| = \frac{|\langle v, (1 - p)w \rangle|}{\text{Tr}(T'(1 - p))} = 0$$

since $w \in \text{Im } p_w \subseteq \text{Im } p = \ker(1 - p)$.

On the other hand, the net $T_p \in B(\mathcal{H})$ does not converge to zero in the ultraweak topology; by linearity of the trace we have that $\text{Tr}(T_p T') = 1$ for all p . We conclude that the ultraweak topology is stronger than the weak operator topology.

We now turn to the case where $A \subseteq B(\mathcal{H})$ is an arbitrary von Neumann subalgebra. Recall from the double commutant theorem that A is ultraweakly closed in $B(\mathcal{H})$. It follows from Theorem 16.4 that it corresponds to a certain subset of $TC(\mathcal{H})^\vee$ closed in the weak*-topology. The fact that A is a von Neumann algebra will follow from very general theory about ‘annihilators’ in Banach spaces. More specifically, we have the following theorem:

Theorem 16.12. *Let X be a Banach space. There is a bijection*

$$\begin{aligned} \{Z \subseteq X \text{ closed}\} &\leftrightarrow \{A \subseteq X^\vee \text{ weak}^* \text{ closed}\} \\ T_2: Z &\mapsto \{f \in X^\vee : f(z) = 0 \forall z \in Z\} \\ \{x \in X : f(x) = 0 \forall f \in A\} &\leftrightarrow A : T_1 \end{aligned}$$

Moreover, evaluation induces isometries of Banach spaces

$$\begin{aligned} \{A \rightarrow \mathbb{C} \text{ linear weak}^* \text{ continuous}\} &\leftarrow X/Z \\ &A \rightarrow (X/Z)^\vee \end{aligned}$$

We will not prove the theorem. The inclusions of the type $A \subseteq T_2T_1A$ are easy. To show the other inclusions such as $A \supseteq T_2T_1A$, you need to apply the Hahn-Banach theorem.

Applying the theorem to the case where $X = TC(\mathcal{H})$, we see that the von Neumann subalgebra $A \subseteq TC(\mathcal{H})^\vee$ corresponds to subspace $Z \subseteq TC(\mathcal{H})$ which is closed in the trace norm. Moreover, X/Z is a predual of A and this predual can be identified with ultraweakly continuous linear functionals on A . In other words, we have proven Theorem 16.3.

Remark 16.13. It also follows from Theorem 16.12 that all ultraweakly continuous linear functionals $A \rightarrow \mathbb{C}$ are of the form

$$a \mapsto \sum_i \langle v_i, aw_i \rangle$$

for some sequence $v_i, w_i \in \mathcal{H}$ such that

$$\sum_i \|v_i\| \|w_i\| < \infty.$$

This statement is already interesting in the case $A = B(\mathcal{H})$.

Example 16.14. Let I be a set and $A = \ell^\infty(I) = \bigoplus_I^{C^*Alg} \mathbb{C}$ the C^* -algebra of bounded functions on I . Then A acts by diagonal operators on the Hilbert direct sum $\ell^2(I) = \bigoplus_I^{Hilb} \mathbb{C}$. We thus get an inclusion $A \hookrightarrow B(\ell^2(I))$. It follows from the analogous case of Lemma 14.2 for arbitrary index sets that $\ell^\infty(I)'' = \ell^\infty(I)$. In fact, one can show that $\ell^\infty(I)' = \ell^1(I)$ by checking on components. We have that $L^1(\ell^\infty(I)) = \ell^1(I)$ is a predual of $\ell^\infty(I)$; the linear map

$$(x_i)_{i \in I} \mapsto \left(y_i \mapsto \sum_i y_i x_i \right)$$

gives an isometry $\ell^\infty(I) \rightarrow (\ell^1(I))^\vee$. One can think of $\ell^1(I)$ as the diagonal trace class operators on $\ell^2(I)$.

Even though a C^* -algebra only had one natural topology, von Neumann algebras have two:

Definition 16.15. A linear map $A_1 \rightarrow A_2$ between von Neumann algebras is *normal* if it is continuous in the weak*-topology coming from $(L^1A_i)^\vee \cong A_i$.

It follows by Theorem 16.3 that a map between von Neumann algebras of $B(\mathcal{H})$ is normal if and only if it is ultraweakly continuous.

Remark 16.16. Since in finite dimensions every operator is trace class, we see that $M_n(\mathbb{C})$ is a predual of $M_n(\mathbb{C})$. Moreover, the ultraweak topology and the operator norm topology agree. In particular \mathbb{C} is a von Neumann algebra and the Euclidean topology agrees with the ultraweak topology. It follows that if $A \subseteq B(\mathcal{H})$ is a von Neumann subalgebra, then normal linear maps $A \rightarrow \mathbb{C}$ are the ultraweakly continuous linear functionals

$$a \mapsto \sum_i \langle v_i, aw_i \rangle$$

discussed before.

Remark 16.17. One can ask the question whether a given map $A_1 \rightarrow A_2$ of von Neumann subalgebras of $B(\mathcal{H})$ is continuous in the weak operator topology. Unlike the ultraweak topology however, this depends on the choice of embedding into $B(\mathcal{H})$. Indeed, by definition of the ultraweak topology we can compose the diagonal embedding $B(\mathcal{H}) \rightarrow B(\mathcal{H}^{\oplus \mathbb{N}})$ to make weak operator continuity into ultraweak continuity, and we know that the WOT and the UWT don't agree.

Remark 16.18. The ultraweak topology is important from the perspective of physics because it is the topology of convergence of observables in expectation values. In more detail, a *density operator* $\rho: \mathcal{H} \rightarrow \mathcal{H}$ is a positive trace class operator such that $\text{Tr}(\rho) = 1$. This induces a so-called (*mixed*) *state* induced by ρ which is the map $B(\mathcal{H}) \rightarrow \mathbb{C}$ given by $a \mapsto \text{Tr}(\rho a)$. The *expectation value* of $T \in B(\mathcal{H})$ with respect to the density ρ is $\text{Tr}(\rho a)$. Then $a_n \rightarrow a$ in the ultraweak topology if and only if $\text{Tr}(\rho a_n) \rightarrow \text{Tr}(\rho a)$ for all density operators ρ . One direction is obvious, for the other direction one has to show that if $\text{Tr}(\rho a_n) \rightarrow \text{Tr}(\rho a)$ for all density operators ρ , then $\text{Tr}(T a_n) \rightarrow \text{Tr}(T a)$ for all trace class operators. This follows from finite rank operators being ultraweakly dense in $B(\mathcal{H})$ since every finite rank operator is a finite linear combination of positive finite rank operators.

17 Continuous functional calculus and positive elements (November 4)

17.1 Continuous functional calculus

In this lecture we will learn of an important application of Gelfand duality to C^* -algebras A that are not necessarily commutative. Namely, we will learn how to make sense of $f(a) \in A$ for continuous functions $f: \text{Spec}(a) \rightarrow \mathbb{C}$ and certain $a \in A$. We emphasize that the domain of f is a subset of \mathbb{C} , but we extend its meaning to allow it to eat elements of A . This is similar to and in fact generalizes the fact that I can take a polynomial p with coefficients in \mathbb{C} and make sense of $p(a)$. It follows from the theory of holomorphic functions in Banach spaces that one can make sense of $f(a)$ in any Banach algebra, but we need to require that f is holomorphic. This is called *holomorphic functional calculus*. The corresponding construction for continuous functions only works for C^* -algebras and is called *continuous functional calculus*.

Theorem 17.1. *Let A be a C^* -algebra and $a \in A$ normal, i.e. $a^*a = aa^*$. Then there exists a unique assignment of an element $f(a) \in A$ to any continuous $f: \text{Spec}(a) \rightarrow \mathbb{C}$ such that*

1. $(\text{id}_{\text{Spec}(a)})(a) = a$
2. $(\lambda f + g)(a) = \lambda f(a) + g(a)$
3. $(f \cdot g)(a) = f(a)g(a)$
4. $\overline{f}(a) = f(a)^*$
5. *The assignment is continuous in f*

for all $\lambda \in \mathbb{C}$ and $f, g: \text{Spec}(a) \rightarrow \mathbb{C}$.

Moreover, this assignment satisfies

(a) $\text{Spec}(f(a)) = f(\text{Spec}(a))$

(b) If $a, b \in A$ commute, and we are given $f: \text{Spec}(a) \rightarrow \mathbb{C}$ and $g: \text{Spec}(b) \rightarrow \mathbb{C}$, then $f(a)$ and $g(b)$ commute.

(c) $(g \circ f)(a) = g(f(a))$ if $g: \text{Spec}(f(a)) \rightarrow \mathbb{C}$

Proof. If any assignment $a \mapsto f(a)$ satisfies 1-4, it is fixed to be the canonical polynomial assignment $a \mapsto p(a)$ on any polynomial in z and \bar{z} with coefficients in \mathbb{C} . Since $\text{Spec}(a)$ is compact, it follows by the Stone-Weierstrass theorem that these polynomials are uniformly dense in $C(\text{Spec}(a))$. It therefore follows by 5 that the assignment is unique.

We will now construct the desired assignment. Most of the construction is an immediate consequence of Gelfand duality.

Let $C^*(a) \subseteq A$ be the smallest C^* -subalgebra that contains a . We can explicitly describe it as the closure of polynomials in a and a^* :

$$C^*(a) = \overline{\text{span}\{a^k(a^*)^l : k, l \geq 0\}}.$$

The order of the a and a^* didn't matter because a is normal. It also follows that $C^*(a)$ is commutative. We therefore know that the Gelfand transform $C^*(a) \rightarrow C(\Phi_{C^*(a)})$ is a $*$ -isomorphism.

We claim that there is a homeomorphism $\Phi_{C^*(a)} \rightarrow \text{Spec}(a)$ given by $\phi \mapsto \phi(a)$. Note that this map is well-defined because $\phi(a) = \hat{a}(\phi) \in \text{Spec}(a)$. It is also surjective since the image of \hat{a} in fact equals $\text{Spec}(a)$. It is continuous by definition of the weak*-topology on $\Phi_{C^*(a)}$. Finally, for injectivity assume that $\phi(a) = 0$ for some $\phi \in \Phi_{C^*(a)}$. Since ϕ is an algebra homomorphism and $\phi(a^*) = \overline{\phi(a)}$, ϕ vanishes on all polynomials in a and a^* . By continuity of ϕ , it vanishes on all of $C^*(a)$. Since every continuous bijection between compact Hausdorff spaces is a homeomorphism, we see that $\Phi_{C^*(a)} \cong \text{Spec}(a)$.

Define $f(a) \in C^*(a)$ to be the image of $f \in C(\text{Spec}(a))$ under

$$C(\text{Spec}(a)) \rightarrow C(\Phi_{C^*(a)}) \rightarrow C^*(a). \quad (11)$$

Concretely, this means that $f(a) \in C^*(a)$ is the unique element such that $\phi(f(a)) = f(\phi(a))$ for all $\phi \in \Phi_{C^*(a)}$.

Points 1-4 are an immediate consequence of the inverse of the composition (11) being a *-homomorphism. Point 5 follows from the fact that the composition $C(\text{Spec}(a)) \rightarrow C^*(a) \rightarrow A$ is a *-homomorphism and hence continuous.

Point (a) follows since

$$\text{Spec}(f(a)) = \text{Spec}(\widehat{f(a)}) = f(\text{Spec}(a)).$$

For point (b), note that the C^* -algebra $C^*(a, b)$ generated by a and b is still commutative and contains the commutative subalgebras $C^*(a)$ and $C^*(b)$. Therefore $f(a) \in C^*(a)$ and $g(b) \in C^*(b)$ commute in $C^*(a, b)$.

Finally, for point (c), first note that $f(a) \in A$ is again normal, so that $C^*(f(a))$ makes sense. Now consider the diagram

$$\begin{array}{ccccc} f(a) & \longleftarrow & (\phi \mapsto f(\phi(a))) & \longleftarrow & f \\ & & & & \\ C^*(a) & \xrightarrow{\sim} & C(\Phi_{C^*(a)}) & \xleftarrow{\sim} & C(\text{Spec}(a)) \\ \uparrow & & \uparrow & & (-) \circ f \uparrow \\ C^*(f(a)) & \xrightarrow{\sim} & C(\Phi_{C^*(f(a))}) & \xleftarrow{\sim} & C(\text{Spec}(f(a))) \end{array}$$

Here the *-homomorphism $C^*(f(a)) \hookrightarrow C^*(a)$ is the unique homomorphism that maps the generator $f(a) \in C^*(f(a))$ to $f(a) \in C^*(a)$. By Gelfand duality, this induces a corresponding *-homomorphism $C(\Phi_{C^*(f(a))}) \rightarrow C(\Phi_{C^*(a)})$ making the left square commute. For the rightmost vertical arrow recall that precomposing with a continuous map indeed gives a *-homomorphism and $\text{Spec}(f(a)) = f(\text{Spec}(a))$.

Showing that $(g \circ f)(a) = g(f(a))$ for all $g \in C(\text{Spec}(f(a)))$ exactly means this diagram commutes. It therefore remains to show that the right square commutes. So let $g \in C(\text{Spec}(f(a)))$. Going through the square eastwards gives the map $\phi \mapsto (g \circ f)(\phi(a))$. Going westwards gives $\phi \mapsto g(\phi(f(a)))$ and $\phi(f(a))$ indeed equals $f(\phi(a))$. This finishes the proof. \square

17.2 Positive elements revisited

Our goal in this subsection is to show the following result.

Theorem 17.2. *Let A be a C^* -algebra and $b \in A$. Then b is positive if and only if it can be written as a^*a for some $a \in A$.*

We first prove a few lemmas.

Lemma 17.3. *If $b \in A$ is self adjoint, then there exists a decomposition $b = b_+ + b_-$ with $b_+ \geq 0, b_- \leq 0$ and $b_+b_- = b_-b_+ = 0$.*

Proof. Define the continuous functions $f_{\pm}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_+(x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases} \quad f_-(x) = \begin{cases} 0 & x \geq 0 \\ x & x \leq 0 \end{cases}$$

Since b is self adjoint, it is normal and $\text{Spec}(b) \subseteq \mathbb{R}$. We can therefore define $b_+ = f_+(b)$ and $b_- = f_-(b)$. Note that $f_+ + f_- = \text{id}_{\mathbb{R}}$ and $f_-f_+ = f_+f_- = 0$. We therefore get

$$b_+ + b_- = f_+(b) + f_-(b) = (f_+ + f_-)(b) = (\text{id}_{\mathbb{R}})(b) = b$$

and similarly $b_+b_- = b_-b_+ = 0$. Since f_+ and $-f_-$ take positive values, they are positive elements of $C(\text{Spec}(b))$ and hence so are $f_+(b)$ and $-f_-(b)$. \square

The following lemma will be an exercise.

Lemma 17.4. *Let $a, b \in A$. Then $\text{Spec}(ab) \setminus \{0\} = \text{Spec}(ba) \setminus \{0\}$.*

We can deduce without the use of continuous functional calculus that elements of the form b^*b can't be negative:

Lemma 17.5. *If $b^*b \leq 0$ then $b = 0$*

Proof. Let $b \in B$. Recall that we can write

$$b = \Re b + i\Im b$$

where

$$\Re b = \frac{b + b^*}{2} \quad \Im b = \frac{b - b^*}{2i}$$

are both self adjoint. We compute

$$(\Re b)^2 + (\Im b)^2 = \frac{bb^* + b^*b}{2}$$

Recall the exercise that the square of a self adjoint element is positive. We have also proven that sums of positive elements are positive. Since clearly positive scalar multiples of positive elements are positive, we see that $bb^* + b^*b \geq 0$ for any $b \in B$.

Now suppose $b^*b \leq 0$. Then follows by Lemma 17.4 that the spectrum of bb^* is also negative. Since bb^* is self adjoint, we therefore obtain that $bb^* \leq 0$. As sums of negative elements are negative, we see that $bb^* + b^*b \geq 0$ and $bb^* + b^*b \leq 0$. By antisymmetry of the order, we conclude that $b^*b = -bb^*$. But then we see that $0 \leq b^*b \leq 0$ and so applying antisymmetry again we obtain $b^*b = 0$. By the C^* -identity $0 = \|b^*b\| = \|b\|^2$ now implies that $b = 0$. \square

Proof of Theorem 17.2. If $b \in A$ is positive, then a is normal and $\text{Spec}(b) \subseteq [0, \infty)$. Therefore, we can apply the square root function $f(x) = \sqrt{x}$ as a map $\text{Spec}(b) \rightarrow \mathbb{R}$ to b . It follows by properties of functional calculus that $\sqrt{b^2} = b$. Since f is real valued, \sqrt{b} is self adjoint and so $a = \sqrt{b}^* \sqrt{b}$.

Conversely, suppose $b = a^*a$. By Lemma 17.3 we can write $b = (a^*a)_+ + (a^*a)_-$ with $(a^*a)_+$ positive and $(a^*a)_-$ negative. We are done once we can show that $(a^*a)_- = 0$.

Define $c := a(a^*a)_-$. Using functional calculus, we can compute

$$c^*c = (a^*a)_- a^* a (a^*a)_- = ((a^*a)_-)^3$$

since $f_-(f_+ + f_-)f_- = f_-^3$. Since the third power of a negative element is negative, we obtain $c^*c \leq 0$ and hence $c = 0$ by Lemma 17.5. But then also

$$0 = a^*c = a^*a(a^*a)_- = ((a^*a)_-)^2 = 0.$$

It follows that $(a^*a)_- = 0$ from the fact that positive elements $a_1 \in A$ have unique negative/positive square roots.

To show this fact, suppose $a_2^2 = a_1$ is a positive square root. Since a_2 commutes with itself, it commutes with a_1 and hence with $\sqrt{a_1}$ by property (b) of functional calculus. Because the C^* -algebra generated by a_2 and $\sqrt{a_1}$ is commutative, we have reduced the problem to the question of whether positive-valued functions on compact Hausdorff spaces have a unique positive square root. This is obviously true. \square

Corollary 17.6. *If $a \in A$ is positive and $b \in A$, then b^*ab is positive.*

Proof. Write $a = c^*c$. Then $b^*ab = (ab)^*ab$ is positive. \square

Remark 17.7. Since $f(z) = |z|$ is continuous on all of \mathbb{C} , we can now make sense of $|a|$ for any normal $a \in A$. More generally, if $a \in A$ is not necessarily normal, we can define $|a| = \sqrt{a^*a}$ using that a^*a is positive and hence has positive spectrum. In particular, we can make sense of $|T|$ for any bounded operator on a Hilbert space \mathcal{H} as we remarked when we discussed trace class operators.

18 Boolean algebras, Stone spaces and Stonean spaces (November 6)

Let A be a commutative C^* -algebra. By Gelfand duality we can write $A = C(X)$ for a unique up to homeomorphism compact Hausdorff space X . We have seen that the poset of projections in A is isomorphic to the poset of clopens in X . This poset has several nice properties; it has a largest element $1 \in A$ (or $X \in \text{clopens}(X)$) and a smallest element $0 \in A$ (or $\emptyset \in \text{clopen}(X)$). It also has suprema and infima for finite sets, corresponding to unions and intersections of clopens respectively. Finally, it has ‘complements’; every projection $p \in A$ (clopen $U \in \text{clopens}(X)$) has a corresponding projection $1 - p \in A$ (clopen $X \setminus U \in \text{clopens}(X)$) satisfying certain properties.

This is saying that $\text{clopens}(X)$ is a bounded complemented distributive lattice, and hence a so-called *Boolean algebra*. Moreover, it turns out that if a commutative von Neumann algebra $C(X)$ has so many projections that this Boolean algebra uniquely determines it.

This is fairly unique; for any connected X , the poset $\text{clopens}(X) = \{\emptyset, X\}$ has only two points. Recall here that X is *connected* if for all decompositions $X = U \sqcup V$ with U, V clopen, we have $U = \emptyset$ or $U = X$. The compact Hausdorff spaces X for which their Boolean algebra of clopens uniquely determines their homeomorphism type are very weird:

Definition 18.1. X is called *Stone* or *totally disconnected* if its only connected subspaces are the singletons.

Equivalently, for all $x, y \in X$ distinct, there exist clopen disjoint neighborhoods U of x and V of y .

Since it seems to be difficult to conceive of reasonable Stone spaces, we give an example.

Example 18.2. The space $X = \mathbb{N} \cup \{\infty\}$ in which opens are the subsets $A \subseteq X$ which are either cofinite or contained in \mathbb{N} . This is the one point compactification of \mathbb{N} . Equivalently, it is the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $[0, 1]$, which makes it clear that it is compact Hausdorff. Note that a map out of this space into some space Y is exactly the same as a convergent sequence in Y .

X is a Stone space; let $x, y \in X$ be distinct. If $x, y \in \mathbb{N}$, then $U = \{x\}$ and $V = \{y\}$ are disjoint and clopen. If one of the points (say y) is equal to ∞ , then $\{x\}$ and $X \setminus \{x\}$ give the desired disjoint clopens.

Remark 18.3. In the literature the term ‘Stone’ usually assumes the space is compact, while a ‘totally disconnected space’ need not be. So for example, the space \mathbb{Q} is totally disconnected but not Stone as it is not compact. Similarly, a discrete space is always totally disconnected, but only Stone if it is finite, since otherwise it is not compact.

To study commutative von Neumann algebras $C(X)$, it will be helpful to understand the Boolean algebra $\text{clopen}(X)$. Moreover, we will see that not any Boolean algebra can be obtained this way; the Boolean algebra will need to have arbitrary suprema and infima and satisfy some further properties. Altogether, this is saying that X is a *hyperstonean space*.

18.1 Boolean algebras

Before coming back to von Neumann algebras and understanding their connection to measure theory, we first study Boolean algebras on their own.

Definition 18.4. A *Boolean algebra* is a ring R such that every element is idempotent, i.e. $x^2 = x$ for all $x \in R$.

Example 18.5. The field \mathbb{F}_2 with two elements. The product is given by ‘and’, and $+$ is given by ‘exclusive or’.

Example 18.6. The other trivial Boolean algebra is $\{0\}$. When comparing with C^* -algebras A (spaces X), you should think of $B = \{0\}$ as $A = \{0\}$ ($X = \emptyset$) and $B = \mathbb{F}_2$ as $A = \mathbb{C}$ ($X = \{\text{pt}\}$). Category-theoretically, $\{0\}$ is a terminal object, and \mathbb{F}_2 an initial object.

Example 18.7. Let R be a commutative ring such as $C(X)$. The set of idempotents is a Boolean algebra with the product of R and addition given by

$$p_1 \oplus p_2 := p_1 + p_2 - p_1 p_2,$$

where on the right we meant the sum in the ring R . It is a straightforward but computationally intensive exercise to verify that this defines a ring structure on the set of idempotents.

In particular, the collection of clopens in X is a Boolean algebra. One can verify that the product is given by intersection and addition is given by the symmetric difference.

Remark 18.8. As we have seen in Example 18.7, the addition in the Boolean algebra plays the role of a symmetric difference. Therefore to avoid confusion with the addition in related structures, it would be better to denote the additive operation by \oplus or Δ instead of $+$ like we are doing.

It is also not immediately obvious to convert results about the C^* -algebra $C(X)$ to its Boolean algebra of projections. It is however clear that if $C(Y) \rightarrow C(X)$ is a $*$ -homomorphism, then it preserves the symmetric difference of projections. So there is still a functor from commutative C^* -algebras¹⁸ to its Boolean algebra of projections.

Proposition 18.9. *Let B be a Boolean algebra*

1. $b = -b$ for all $b \in B$
2. B is commutative.

Proof. For point 1 we compute

$$2b = (2b)^2 = 4b^2 = 4b$$

and so $2b = 0$. For point 2, we compute for $b_1, b_2 \in B$ that

$$b_1 + b_2 = (b_1 + b_2)^2 = b_1^2 + b_1 b_2 + b_2 b_1 + b_2^2 \implies b_1 b_2 = -b_2 b_1,$$

so done by point 1. □

It follows from the above proposition that every Boolean algebra is an algebra over \mathbb{F}_2 . We will however not use this as I have promised at the beginning of the course that all algebras will be over \mathbb{C} .

Next we will show that every Boolean algebra has the structure of a poset. As a motivation, we will prove the following proposition.

Proposition 18.10. *Let A be a C^* -algebra (not necessarily commutative) and let $p, p' \in A$ be projections. The following are equivalent*

1. $p' \geq p$
2. $p' p = p$

¹⁸Or from the opposite of compact Hausdorff spaces, or commutative rings with ring homomorphisms.

3. $pp' = p$.

Proof. If $p' - p \geq 0$, we first prove that $p = pp'p$. Conjugate by the self adjoint p and use Corollary 17.6 to get that

$$p(p' - p)p = pp'p - p \geq 0.$$

On the other hand, we have that

$$p - pp'p = p(1 - p')(p(1 - p'))^* \geq 0.$$

Hence $pp'p - p = 0$. Now we compute

$$(p - p'p)^*(p - p'p) = p - pp'p = p - p = 0$$

and conclude $p = p'p$. Completely analogously we prove $p = pp'$.

Conversely, suppose $p = p'p$. Then we get $p = pp'$ by taking adjoints. Hence

$$(1 - p)p'(1 - p) = p' - pp' - p'p + pp'p = p' - p \geq 0$$

since $p' \geq 0$. □

Definition 18.11. A *supremum* $x \in X$ of a collection of elements $\{x_i\}_{i \in I}$ in a poset X is a least upper bound, i.e. $x \in X$ such that $x_i \leq x$ for all $i \in I$ and if $y \in X$ satisfies $x_i \leq y$ for all i , then $x \leq y$. The *infimum* is similar but the other way.

In a general poset, suprema and infima might not exist, but if they do they are unique¹⁹.

Proposition 18.12. *Let B be a Boolean algebra. Then $x \leq y \iff x = xy$ makes B into a poset with largest element 1 and smallest element 0. Moreover*

1. $xy = \inf\{x, y\}$
2. $x + y + xy = \sup\{x, y\}$

Proof. Reflexivity of the relation follows as $x^2 = x$. For antisymmetry, if $x \leq y$ and $y \leq x$, we have $x = xy = y$. Finally, if $x \leq y$ and $y \leq z$, then $xz = xyz = xy = x$, so B is transitive. We have proven that B is a poset. It is clear that $x \leq 1$ and $0 \leq x$ for all x .

To show that $xy = \inf\{x, y\}$, we have to prove that $xy \leq x$ and $xy \leq y$ (which is obvious), as well as that it is the largest such lower bound of x and y . So let $z \leq x$ and $z \leq y$. We have to show that $xy \geq z$. This follows from $zxy = zy = z$.

Finally we have to show that $x + y + xy$ is the smallest upper bound of $\{x, y\}$. For this we compute

$$x(x + y + xy) = x + xy + xy = x$$

so that $x \leq x + y + xy$ and similarly for y . Now suppose that $z \geq x$ and $z \geq y$. We then deduce $z(x + y + xy) = x + y + xy$ and so $z \geq x + y + xy$ as desired. □

We can conclude from the above proposition that Boolean algebras have finite suprema and finite infima.

¹⁹In fact, a poset P gives a category with objects P in which there is exactly one morphism $x \rightarrow y$ when $x \leq y$ and none otherwise. The product in this category is the infimum and the coproduct is the supremum.

Definition 18.13. The *complement* of $b \in B$ is $b^\perp = 1 - b$.

Note that $\inf\{b, b^\perp\} = 0$ and $\sup\{b, b^\perp\} = b + b^\perp + bb^\perp = 1$.

Definition 18.14. A poset P is called *totally ordered* if for all $x, y \in P$ we have $x \leq y$ or $y \leq x$.

In a totally ordered set, the infimum of two elements is always given by their minimum. In a general poset this need not be the case, as the minimum of two elements need not make sense.

Example 18.15. If S is a set, then $B = P(S) \cong \text{Map}(S, \mathbb{F}_2)$ is a Boolean algebra with pointwise addition and product. From the perspective of $P(S)$, the product is given by intersection, the sum by symmetric difference, the order by inclusion, $1 = S$ and $0 = \emptyset$. An ideal $I \subseteq P(X)$ is a nonempty downward closed directed subset. The supremum of two elements is given by their union. Note that B even has arbitrary suprema given by unions (and arbitrary infima given by intersections).

If S has cardinality at least 2, this poset is not totally ordered. Indeed, if $x, y \in S$ are different, then $\{x\}$ and $\{y\}$ are not comparable. Note also that we have $\sup\{\{x\}, \{y\}\} = \{x, y\}$ and $\inf\{\{x\}, \{y\}\} = \emptyset$. So in general the infimum/supremum of a finite number of elements doesn't need to be the minimum/maximum of those elements.

Remark 18.16. In the literature on order theory, the infimum is usually written as \wedge and pronounced 'meet' and the supremum is written \vee and pronounced 'join'. This notation is more reasonable in $P(S)$ or other Boolean algebras in which elements are subsets and order is inclusion, such as $\text{clopen}(X)$. The notation also makes formulas such as (12) look more reasonable;

$$a + b = (a \vee b) \wedge (a \vee b)^\perp.$$

This notation is also less confusing in the context of logic. For example, thinking of $B = \mathbb{F}_2$ as truth values, the supremum of two elements is given by 'or'.

Generalizing Example 18.15, there are actually very few totally ordered Boolean algebras:

Example 18.17. Let B be a totally ordered Boolean algebra and let $x \in B$. Since $\inf\{x, x^\perp\} = 0$, we know that $x = 0$ or $x^\perp = 0$. Therefore $B = 0$ or $B = \mathbb{F}_2$.

One can also show that infima and suprema distribute over each other in a Boolean algebra. The converse is also true, but we won't prove it:

Theorem 18.18. *Let P be a poset with largest element 1, smallest element 0 and finite infima and suprema which distribute over each other (i.e. a distributive lattice). If P admits complements in the sense of $x^\perp \in P$ such that $\inf\{x, x^\perp\} = 0$ and $\sup\{x, x^\perp\} = 1$, then P defines a Boolean algebra where the addition is given by the symmetric difference*

$$a + b = \inf\{\sup\{a, b\}, \inf\{a, b\}^\perp\} \tag{12}$$

To go back to spaces from Boolean algebras, we take the Zariski spectrum with the Zariski topology. We therefore study the ideals of Boolean algebras, with particular interest in the maximal ones.

Proposition 18.19. *A subset $I \subseteq B$ is an ideal if and only if*

1. $I \neq \emptyset$
2. I is downward closed, i.e. if $y \in I$ and $x \leq y$, then $x \in I$

3. Any two elements in I have a common upper bound in I , i.e. for all $x, y \in I$ there is a $z \in I$ such that $x \leq z$ and $y \leq z$

Proof. Let $I \subseteq B$ be an ideal. Since $0 \in I$, we get the first point. For the second point, if $x \leq y$ for some $y \in I$, then $x = xy \in I$. Finally, for the third point, if $x, y \in I$, then $\sup\{x, y\} = x + y + xy \in I$ is a common upper bound for x and y .

Now assume $I \subseteq B$ satisfies conditions 1-3. If $x \in I$ and $y \in B$, then $xy \leq x$ and so $xy \in I$ by 2.

Finally, we have to check that if $x, y \in I$, then their sum is in I . Pick $z \in I$ such that $z \geq x$ and $z \geq y$. Then by point 2 and the fact that $\sup\{x, y\} \leq z$, we get that $x + y + xy \in I$. But $x + y + xy \geq x + y$ since

$$(x + y + xy)(x + y) = x + xy + yx + y + xy + xy = x + y$$

so by point 2 again we get $x + y \in I$. □

Proposition 18.20. *The following are equivalent for an ideal $I \subseteq B$.*

1. I is maximal
2. I is prime
3. for all $a \in B$ either $a \in I$ or $a^\perp \in I$ but not both.

Proof. We always have $1 \implies 2$. We prove $2 \implies 3$. Let $I \subseteq B$ be a prime ideal. If $a \in B$, then $aa^\perp = 0 \in I$ and so $a \in I$ or $a^\perp \in I$. If both would be in I , then $a + a^\perp = 1 \in I$ and so $I = B$, contradicting the fact that I is prime.

Finally we show $3 \implies 1$. Let $I \subseteq B$ be an ideal satisfying 3. Note that $I \neq B$ since $0 \in I$ and so $1 \notin I$. Suppose $I \subseteq J$ is a strictly larger ideal. Let $a \in J$ not be in I . Then $a^\perp \in I$ and so $a^\perp \in J$. But then $a + a^\perp = 1 \in J$ and so $J = B$. □

Definition 18.21. Let B be a Boolean algebra. The *Pierce spectrum* of B is the Zariski spectrum $\text{Zar } B$ with the Zariski topology.

Remark 18.22. Because of Proposition 18.20, the maximal ideal spectrum of B is the same as the prime ideal spectrum of B . This is convenient because the prime ideal spectrum has good functoriality properties. In other words, a homomorphism of rings $B_1 \rightarrow B_2$ induces a continuous map $\text{Zar}(B_2) \rightarrow \text{Zar}(B_1)$.

Remark 18.23. The above definition is slightly nonstandard as it is usually made using ‘ultrafilters’ instead of ideals. Filters are simply complements of ideals and ultrafilters correspond to maximal ideals.

Recall that the Zariski topology is generated by the basis of opens

$$U_b = \{I \subseteq B \text{ prime ideal} : b \notin I\}$$

The reason this is a basis is that $U_{b_1} \cap U_{b_2} = U_{b_1 b_2}$. Indeed, for prime ideals we have that $b_1 b_2 \notin I$ if and only if $b_1 \notin I$ and $b_2 \notin I$.

Clearly $U_1 = B$. Conversely, if $U_b = B$, then b is not in any maximal ideal of B . But then (b) , the ideal generated by b can’t be proper, since every proper ideal is contained in a maximal ideal. Hence there is a $x \in B$ such that $xb = 1$. But then $\inf\{b, x\} = 1$ is the greatest element of B and so $b = 1$.

It follows by Proposition 18.20 that $U_{b^\perp} = B \setminus U_b$ and so in particular all basic opens are clopen. This does not mean in general that all open sets are clopen, since opens are arbitrary unions of basic clopens, which need not be closed anymore. We therefore also have $U_b = \emptyset$ if and only if $b = 0$.

Proposition 18.24. *If B is a Boolean algebra, then $\text{Zar } B$ is a Stone space.*

Proof. For any commutative ring R the Zariski spectrum $\text{Zar } R$ is compact. Let $I, J \in \text{Zar } B$ be distinct. Pick $b \in I$ not in J . Then $J \in U_b$ and $I \notin U_b$, i.e. $I \in U_{b^\perp}$. By Proposition 18.20, we have that $\text{Zar } B = U_b \sqcup U_{b^\perp}$. So J and I can be separated by disjoint clopens. It follows that $\text{Zar } B$ is Hausdorff, as well as Stone. \square

Remark 18.25. It might give one less of a headache to work with the clopens $\text{Zar } B \setminus U_b = \{I \subseteq B \text{ maximal} : b \in I\}$ instead of U_b , because the condition $b \notin I$ is more confusing.

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