Operator algebra course

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Contents

1 Introduction Lecture (September 2)

Example 1.1. Let $n \geq 0$ be an integer. $M_n(\mathbb{C})$ is the collection of n by n matrices with entries in $\mathbb{C}$. From linear algebra we know matrices can be added, multiplied by scalars and we have matrix multiplication with the identity matrix as the unit. There are several matrix norms but one standard one is

$$\|A\| = \sup_{v \in \mathbb{C}^n \setminus 0} \frac{\|Av\|_{\mathbb{C}^n}}{\|v\|_{\mathbb{C}^n}},$$

where $\|v\|_{\mathbb{C}^n}$ is the standard Euclidean norm $|v_1|^2 + \cdots + |v_n|^2$. There is a notion $A^* = (\overline{a_{ji}})_{i,j}$ of a conjugate transpose of a matrix $A = (a_{ij})_{i,j}$.

Example 1.2. Let $\ell^\infty(\mathbb{N})$ be the space of bounded sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{C}$. These can be entrywise added and multiplied and the constant sequence 1 is a unit. They also have a norm

$$\|(x_n)\| = \sup_n |x_n|.$$

Moreover, this makes $\ell^\infty(\mathbb{N})$ into a Banach space, i.e. a complete normed vector space. Sequences also have a conjugate operation $(x_n)_n^* = (\overline{x_n})_n$.

Example 1.3. Let $C([0,1])$ be the space of continuous complex valued functions on a closed interval. These can be added, multiplied and complex conjugated. Since continuous functions on $[0,1]$ are bounded, the norm

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

is finite.

Definition 1.4. An *algebra* is a complex vector space A equipped with a bilinear multiplication map $\cdot \colon A \times A \to A$ which is associative and has a unit $1 \in A$.

Example 1.5. The set $\mathbb{C}[x]$ of polynomials in a since variable x is an algebra. Even though polynomial algebras have a rich geometric theory (algebraic geometry), this is not the type of algebras we consider in this course. Instead we will consider algebras which are more analytic in nature. They will be 'larger'; they are complete normed spaces.

Definition 1.6. A *normed vector space* is a complex vector space X equipped with a *norm* $\|.\| \colon X \to \mathbb{R}_{\geq 0}$ satisfying

$$\|x + y\| \leq \|x\| + \|y\| \quad \|\lambda x\| = |\lambda|\|x\|$$

and $\|x\| = 0$ if and only if $x = 0$.

Every normed vector space X defines a metric space with $d(x,y) = \|x - y\|$.

Definition 1.7. Recall that a *topology* on a set X is a collection $\mathcal{T}$ of subsets of X called *open sets* which are closed under finite intersections and arbitrary unions such that $\emptyset \in \mathcal{T}$. A *closed set* is the complement of an open set. A set with a topology on it is called a *(topological) space*. We say that the topology $\mathcal{T}'$ is *stronger* than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$.

Example 1.8. If (M,d) is a metric space, then $\mathcal{T}$ given by unions of open balls defines a topology on M. In other words, open balls form a *basis* for the topology on M.

Definition 1.9. A(n open) *neighborhood* of $x \in X$ is an open set containing x.

Exercise 1. A subset $U \subseteq X$ is open if and only if every $x \in U$ has an open neighborhood contained in U.

Definition 1.10. A sequence x_n in a topological space *converges to* x if for every open neighbourhood U of x there exists an N such that $x_n \in U$ for all $n > N$.

Exercise 2. A sequence x_n in a metric space *converges to* x if and only if for all $\epsilon > 0$ there exists an N such that $\forall n > N$ we have $d(x, x_n) < \epsilon$.

Definition 1.11. A sequence in a metric space is *Cauchy* if for all $\epsilon > 0$ there exists an N such that $d(x_n, x_m) < \epsilon$ if $n, m > N$. A metric space is *complete* when every Cauchy sequence converges to some element.

Many topologies in analysis can be studied very well using sequences because they are first countable.

Definition 1.12. We say that X is *first countable* if every point $x \in X$ has a countable neighbourhood basis, i.e. there exists a countable collection U_n of neighborhoods of x such that every open neighborhood of x contains U_n for some n.

Example 1.13. Any metric space is first countable with U_n the balls of radius $1/n$.

Lemma 1.14. *A subset A of a first countable space X is closed if and only if every sequence a_n in A which converges in X also converges in A.*

Proof. '$\implies$' This direction holds in general. Suppose A is closed and a_n converges to $x \in X$. Let us assume $x \in X \setminus A$ and work towards a contradiction. Because $X \setminus A$ is open, there exists an open neighborhood U of x disjoint from A. But by convergence we have that $a_n \in U$ for sufficiently large n, a contradiction.

'$\impliedby$' We prove the contrapositive and so assume $X \setminus A$ is not open. Then there is some $x \in X \setminus A$ such that every open neighbourhood U of x intersects nontrivially with A. Pick a countable neighborhood basis U_n of x. Pick an arbitrary $a_n \in U_n \cap A$ for every n. Because U_n is a neighborhood basis, a_n is a sequence of elements of A converging to $x \in X \setminus A$. □

Definition 1.15. A *Banach space* is a normed vector space which is complete.

Example 1.16. A closed subspace Y of a Banach space X is a Banach space. Indeed, a Cauchy sequence in Y converges in X and its limit is in Y by Lemma 1.14.

Example 1.17. Let $p \geq 1$. Let ℓ^p be the collection of sequences x_n such that $\sum_n |x_n|^p < \infty$. It can be shown to be a Banach space with norm

$$\|x_n\| = \left(\sum_n |x_n|^p \right)^{1/p}.$$

Definition 1.18. A *Banach algebra* is an algebra that is also a Banach space such that $\|ab\| \leq \|a\|\|b\|$. A C^**-algebra* is a Banach algebra equipped with a 'star' operation $*\colon A \to A$ such that

$$a^{**} = a \quad (ab)^* = b^* a^* \quad (a+b)^* = a^* + b^* \quad (\lambda a)^* = \overline{\lambda} a^*$$

as well as the C^**-identity*

$$\|a^* a\| = \|a\|^2.$$

Remark 1.19.

1. The C^*-identity is somewhat mysterious. We will prove all sorts of nice consequences of it next week. For example, the norm is completely determined by the algebraic structure. You will also see how some Banach algebras are 'less nice' in the exercises.

2. There are obvious notions of algebra over $\mathbb{R}$, Banach space over $\mathbb{R}$ and Banach algebra over $\mathbb{R}$. However, the correct definition of C^*-algebra over $\mathbb{R}$ is rather subtle and will not be considered in this course.

3. The definition of Banach algebra and C^*-algebra also make sense without a unit. In fact, most references don't assume unitality because there are many examples that don't satisfy it. However, we will not study non-unital algebras because they will give you a headache.

Example 1.20. The algebra $\mathbb{C}$ with $\|z\| := |z|$ and complex conjugation is a C^*-algebra.

Example 1.21. Let $\{A_i\}_{i\in I}$ be a family of C^*-algebras. Define

$$\bigoplus_i A_i = \{(a_i)_{i\in I} : \sup_{i\in I} \|a_i\| < \infty\}.$$

This is an algebra with pointwise sum, product and scalar multiplication. It has a pointwise $*$ and obvious norm which makes it into a C^*-algebra. In particular, $\ell^\infty(S) := \bigoplus_{s\in S} \mathbb{C}$ is a C^*-algebra for a set S.

Exercise 3. Show that $A = M_2(\mathbb{C})$ with $\|a_{ij}\| := \sum_{i,j} |a_{i,j}|$ is a Banach algebra .

Definition 1.22. A map $f\colon X \to Y$ between topological spaces is *continuous* if for all $U \subseteq Y$ open, $f^{-1}(U)$ is open.

Exercise 4. A map between metric spaces is continuous if and only if it satisfies the usual epsilon delta definition.

Definition 1.23. A space X is *compact* if for all open covers of X — i.e. a family of opens U_i for $i \in I$ such that $\cup_{i\in I} U_i = X$ — there exists a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$.

Lemma 1.24. *Let X be a compact topological space. Then every continuous function $f\colon X \to \mathbb{C}$ is bounded.*

Proof. Let $f\colon X \to \mathbb{C}$ be continuous. For every $r \in \mathbb{R}$, define the open sets $U_r = \{x \in X : |f(x)| < r\} = f^{-1}(B_r)$. Here B_r is the open disk of radius r in the complex plane. Note that $\bigcup_r U_r = X$ is an open cover of X. Because X is compact, there is a finite subcover $\{U_{r_1}, \dots, U_{r_n}\}$. We see that $|f(x)| \leq \max\{r_1, \dots, r_n\}$ for all $x \in X$. □

Definition 1.25. If A is a C^*-algebra, we say that a subspace $B \subseteq A$ is a *C^*-subalgebra* if it is closed in the norm topology, as well as closed under multiplication, the $*$ operation and $1 \in B$.

Since a closed subspace of a Banach space is a Banach space, a C^*-subalgebra is again a C^*-algebra.

Proposition 1.26. *Let X be a compact topological space. Then $C(X)$ is a C^*-algebra.*

Proof. We have seen that $\ell^\infty(X)$ is a C^*-algebra. In Lemma 1.24, we have seen $C(X)$ is a subset of $\ell^\infty(X)$. It therefore suffices to show that it is a closed unital $*$-subalgebra. Since complex conjugation, addition and multiplication preserves continuity, and the constant function 1 is continuous, this is a $*$-subalgebra.

It remains to show that $C(X) \subseteq \ell^\infty(X)$ is closed. Because norm spaces are first countable, it suffices to show that for every sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions with limit $f \in \ell^\infty(X)$, the limit is also continuous. This follows by the uniform limit theorem. The short proof on wikipedia is copied here: we have to show that for every $\epsilon > 0$, there exists a neighborhood U of any point $x \in X$ such that:

$$|f(x) - f(y)| < \varepsilon, \qquad \forall y \in U$$

Consider an arbitrary $\epsilon > 0$. Since the sequence of functions f_n converges uniformly to f by hypothesis, there exists a natural number N such that:

$$|f_N(t) - f(t)| < \frac{\varepsilon}{3}, \qquad \forall t \in X$$

Moreover, since f_N is continuous, for every x there exists a neighborhood U such that:

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}, \qquad \forall y \in U$$

In the final step, we apply the triangle inequality in the following way:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| && (1)\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \qquad \forall y \in U. && (2) \end{aligned}$$

□

Next lecture, we will show continuous operators on a Hilbert space form a C^*-algebra. After that we will need to develop some general theory with the goal of proving an equivalence of categories between compact Hausdorff spaces and commutative C^*-algebras in a few weeks. After that we will move on to von Neumann algebras.

2 Operators on Hilbert space (September 4)

Today we will discuss the main example of a C^*-algebra: continuous operators on a Hilbert space.

We start defining operators on Banach spaces, which already form a Banach algebra. A linear map between infinite-dimensional Banach spaces is not always continuous, but it turns out to be continuous exactly when the following straightforward condition holds.

Definition 2.1. A linear map between Banach spaces $T: X \to Y$ (also called an *operator*) is *bounded* if there exists a $C > 0$ such that $\|Tx\| \le C\|x\|$ for all $x \in X$.

Clearly only the zero map is bounded in the literal sense, so the terminology should not be too confusing.

Definition 2.2. Let $B(X,Y)$ denote the bounded operators $X \to Y$ and $B(X) := B(X,X)$. The *operator norm* on $B(X,Y)$ is defined by

$$\|T\| := \sup_{x \in X\setminus\{0\}} \frac{\|T(x)\|}{\|x\|}$$

The quotient is defined because $\|x\| \neq 0$ if $x \neq 0$. The norm is finite as $\|T\| \le C$, where C is the bound in the definition of a bounded linear map. Moreover, if $x \neq 0$, then

$$\frac{\|Tx\|}{\|x\|} \le \sup_{y \in X\setminus\{0\}} \frac{\|Ty\|}{\|y\|} = \|T\| \implies \|Tx\| \le \|T\|\|x\|$$

and the equation also clearly holds when $x = 0$.

If $T, S \in B(X)$ and $\lambda \in \mathbb{C}$, then $\|\lambda T\| = |\lambda|\|T\|$ and by the triangle equality in X

$$\|T+S\| = \sup_{x \in X\setminus\{0\}} \frac{\|Tx+Sx\|}{\|x\|} \le \sup_{x \in X\setminus\{0\}} \frac{\|Tx\|+\|Sx\|}{\|x\|} \le \sup_{x \in X\setminus\{0\}} \frac{\|Tx\|}{\|x\|} + \sup_{y \in X\setminus\{0\}} \frac{\|Ty\|}{\|y\|} = \|T\|+\|S\|.$$

We see that $B(X)$ is a normed space.

Note that $\|\operatorname{id}_X\| = 1$ is bounded. Also note that

$$\|TS\| = \sup_{\|x\|=1} \|TSx\| \le \sup_{\|x\|=1} (\|T\|\|Sx\|) = \|T\|\|S\|.$$

We conclude that $B(X)$ is a Banach algebra as long as we can prove:

Lemma 2.3. *$B(X,Y)$ is complete.*

Proof. Let T_n be a Cauchy sequence. The first claim is that for all $x \in X$, the sequence $T_n x$ converges. Indeed, let $x \in X$ and note that the result is trivial unless $x \neq 0$. Now take N so that $\|T_m - T_n\| < \epsilon/\|x\|$. Then

$$\|T_m x - T_n x\| \le \|T_m - T_n\|\|x\| < \epsilon$$

and so $T_m x$ is Cauchy. By completeness of Y, $T_m x$ converges to some element of X we will call Tx.

To finish, we need to show that T is a bounded linear operator and T_n converges to T. If $x_1, x_2 \in X$, then

$$T(x_1) + T(x_2) = \lim(T_n(x_1) + T_n(x_2)) = T(x_1 + x_2)$$

by linearity of all T_n and the fact that sums of convergent sequences converge to the sum of the limit (continuity of addition). This holds similarly for scalar multiplication, and so T is linear.

Let $\epsilon > 0$. It suffices to show that $\|T - T_n\| < \epsilon$ for n sufficiently large. Indeed, then T is automatically bounded since $T - T_n$ is bounded. Pick N so that $\|T_m - T_n\| < \epsilon/2$ for all $n, m > N$. We claim that $\|T - T_n\| < \epsilon$ if $n > N$. Let $x \in X$. By definition of Tx, we can pick M so that $\|T(x) - T_n(x)\| < \epsilon\|x\|/2$ for all $n > M$. Then if $m > \max\{N, M\}$, we have that

$$\|(T - T_n)(x)\| \leq \|Tx - T_m x\| + \|T_m x - T_n x\| \leq \|Tx - T_m x\| + \|T_m - T_n\|\|x\| < \epsilon\|x\|.$$

We see that $\|T - T_n\| < \epsilon$ for all $n > M$. □

Sometimes the formula

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

is convenient, which we prove as follows. By restricting the supremum to a subset, we have

$$\sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} \geq \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|T(x)\|.$$

Conversely note that when $x \neq 0$

$$\|Tx\| = \left\| \|x\| T\left(\frac{x}{\|x\|}\right) \right\| = \|x\|\|T(x/\|x\|)\|$$

Since $\|x/\|x\|\| = 1$, we thus get

$$\frac{\|Tx\|}{\|x\|} = \|T(x/\|x\|)\| \leq \sup_{\|y\|=1} \frac{\|T(y)\|}{\|y\|},$$

finishing the argument.

Definition 2.4. An *inner product space* is a vector space $\mathcal{H}$ equipped with an inner product. Here an *inner product* is a sequilinear map

$$\langle -, - \rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

such that

1. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in \mathcal{H}$ (hence $\langle v, v \rangle \in \mathbb{R}$)
2. $\langle v, v \rangle \geq 0$ for all $v \in \mathcal{H}$
3. $\langle v, v \rangle = 0$ if and only if $v = 0$.

By *sesquilinear*, we mean that

$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \overline{\lambda_1}\langle v_1, w \rangle + \overline{\lambda_2}\langle v_2, w \rangle \quad \langle v, \mu_1 w_1 + \mu_2 w_2 \rangle = \mu_1 \langle v, w_1 \rangle + \mu_2 \langle v, w_2 \rangle.$$

Warning 2.5. Many authors assume instead that inner products are complex linear in the first argument and complex anti-linear in the second, instead of the other way around.

We want to be able to talk about complete inner product spaces. By complete, we will mean that the induced normed space $\|v\| := \sqrt{\langle v, v\rangle}$ is a Banach space. Most of the axioms of a normed space are easy to check, but the triangle equality is not immediately clear. It can be shown as a consequence of the following important lemma (exercise).

Lemma 2.6 (Cauchy–Schwarz)**.** *For all $x, y \in \mathcal{H}$ we have that*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Proof. You can find one on Wikipedia. □

Definition 2.7. A *Hilbert space* is a (complex) vector space $\mathcal{H}$ with a complete inner product.

Example 2.8. $\mathbb{C}$ is a Hilbert space with $\langle z_1, z_2\rangle = \overline{z_1} z_2$.

Definition 2.9. A *(bounded) (linear) functional* on a Banach space X is a bounded linear map $X \to \mathbb{C}$. Given a Banach space X, let $X^* := B(X, \mathbb{C})$ be the Banach space of bounded linear functionals.

Example 2.10. Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{H}$. The functional $\langle x, -\rangle \colon \mathcal{H} \to \mathbb{C}$ is bounded of operator norm $\|x\|$. In other words

$$\|x\| = \sup_{y \in \mathcal{H}} \frac{|\langle x, y\rangle|}{\|y\|}.$$

Since this is obvious for $x = 0$, we assume $x \neq 0$. Indeed, by Cauchy-Schwarz we have that

$$\|x\| \leq \frac{|\langle x, y\rangle|}{\|y\|}$$

for all nonzero $y \in \mathcal{H}$. Moreover, the supremum is attained by $y = x$ as

$$\|x\| = \frac{\langle x, x\rangle}{\|x\|}.$$

The following lemma requires some work to prove.

Lemma 2.11. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces If $T \colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator, there exists a unique operator $T^* \colon \mathcal{H}_2 \to \mathcal{H}_1$ such that*

$$\langle Tv, w\rangle = \langle v, T^* w\rangle$$

for all $v \in \mathcal{H}_1$ and $w \in \mathcal{H}_2$.

Assuming the lemma, however, it is not that difficult to show:

Theorem 2.12. *Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ is a C^*-algebra.*

Proof. We have already shown $B(\mathcal{H})$ is a Banach algebra. By using the uniqueness property of the adjoint, we see that $(T + S)^* = T^* + S^*$ and $(\lambda T)^* = \overline{\lambda} T^*$ by sesquilinearity of the inner product. Also note that $T^{**} = T$ since

$$\langle T^* v, w\rangle = \overline{\langle w, T^* v\rangle} = \overline{\langle Tw, v\rangle} = \langle v, Tw\rangle.$$

Finally $(ST)^* = T^*S^*$ is also clear.

So it remains to show the C^*-identity. For this, first note that by Example 2.10, we have for any operator T that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1,\|y\|=1} |\langle Tx, y\rangle|.$$

Applying this to T^*T, we see that

$$\|T^*T\| = \sup_{\|x\|=1,\|y\|=1} |\langle T^*Tx, y\rangle| = \sup_{\|x\|=1,\|y\|=1} |\langle Tx, Ty\rangle|.$$

Now we have on the one hand that

$$\sup_{\|x\|=1,\|y\|=1} |\langle Tx, Ty\rangle| \geq \sup_{\|x\|=1} |\langle Tx, Tx\rangle| = \sup_{\|x\|=1} \|Tx\|^2 = \left(\sup_{\|x\|=1} \|Tx\|\right)^2 = \|T\|^2$$

while on the other hand

$$\sup_{\|x\|=1,\|y\|=1} |\langle Tx, Ty\rangle| \leq \sup_{\|x\|=1,\|y\|=1} \|Tx\|\|Ty\| = \|T\|^2.$$

□

For the proof of the existence of adjoints, we need a classical theorem of Hilbert space theory.

Theorem 2.13 (Riesz-Fréchet)**.** *The map $RF\colon \mathcal{H} \to \mathcal{H}^*$ defined by sending x to $\langle x, -\rangle$ is a complex antilinear isomorphism of Banach spaces.*

Here a map $T\colon X \to Y$ is called *complex antilinear* if $T(x_1+x_2) = Tx_1+Tx_2$ and $T(\lambda x) = \overline{\lambda}Tx$. It still makes sense to talk about boundedness for such maps.

The fact that RF is complex antilinear follows from sesquilinearity of the inner product. It follows from Exercise 2.10 that the RF is an *isometry*, i.e. $\|Tx\| = \|x\|$. An isometry is in particular bounded. Injectivity then follows because every isometry is injective. So the surjectivity, i.e. the fact that every bounded functional $f\colon \mathcal{H} \to \mathbb{C}$ is of the form $\langle x, -\rangle$ for some $x \in \mathcal{H}$ is the nontrivial part. This fact is proven in[1] [1, Number 5] using the theory of projections.

Now, given the Riesz-Fréchet theorem, we can define T^*x as the inverse of the functional

$$y \mapsto \langle x, Ty\rangle,$$

under the Riesz-Fréchet isomorphism. In other words, uniqueness follows by injectivity of RF and existence by surjectivity. We can conclude that $B(\mathcal{H})$ is a C^*-algebra.

If I is a (potentially uncountable) set and $(x_i)_{i\in I}$ a collection of elements in a Banach space X, we say that $\sum_{i\in I} x_i$ *converges to the value* $x \in X$ if for every $\epsilon > 0$ there exists a finite set $F_0 \subseteq I$ such that $\|\sum_{i\in F} x_i - x\| < \epsilon$ whenever F contains F_0.

Example 2.14. Let $\mathcal{H}_i$ for $i \in I$ be a family of Hilbert spaces. Then the *direct sum Hilbert space* is

$$\bigoplus_{i\in I} \mathcal{H}_i = \{v_i \in \mathcal{H}_i : \sum_{i\in I} |v_i|^2 < \infty\}$$

[1] The section numbering in Bram's PhD thesis is potentially confusing. There are sections, subsections and subsubsections as usual. But deeper than that there are also 'numbers' which are then divided into roman numerals which I will refer to as 'paragraphs'. So Number 3 Paragraph V refers to the definition of direct sum of C^*-algebras.

with inner product

$$\langle (v_i)_{i\in I}, (w_i)_{i\in I}\rangle := \sum_{i\in I} \langle v_i, w_i\rangle.$$

This is shown in [1, Number 6 Paragraph II]. As a subexample, we can take all $\mathcal{H}_i$ to be $\mathbb{C}$. We obtain that the Banach space $\ell^2(S)$ is a Hilbert space with inner product

$$\langle (x_n), (y_n)\rangle = \sum_n \overline{x_n} y_n.$$

3 The category of C^*-algebras (September 9)

Definition 3.1. Let A be a C^*-algebra. An element $a \in A$ is called

1. *invertible* if there exists a (necessarily unique) $a^{-1} \in A$ such that $a^{-1}a = 1 = aa^{-1}$.
2. *self-adjoint* if $a^* = a$
3. *unitary* if a^* is the inverse of a

Example 3.2. For $A = M_n(\mathbb{C})$, this reproduces the notions of self adjoint and unitary matrix you already know.

Example 3.3. In $C(X)$, we have that $f \in C(X)$ is self adjoint if and only if it is real valued. A function is unitary if and only if it is valued in the unit circle.

Definition 3.4. Let A, B be C^*-algebras. A *$*$-homomorphism* $\phi\colon A \to B$ is a (unital) algebra homomorphism such that $\phi(a^*) = \phi(a)^*$.

Remark 3.5. It might be surprising that we didn't require ϕ to be bounded. The reason is that this will be automatic, but we don't have the machinery to prove this yet.

We do a small detour reviewing the basics of category theory. This will be useful later to express Gelfand duality. Indeed, Gelfand duality will say how commutative C^*-algebras are 'the same' as compact Hausdorff[2] topological spaces, but how to we express this?

Definition 3.6. A *category* $\mathcal{C}$ consists of a class $\operatorname{ob}\mathcal{C}$ called *objects*, for all $x, y \in \operatorname{ob}\mathcal{C}$ a set $\operatorname{Hom}_{\mathcal{C}}(x, y)$ of *morphisms* and for all $x, y, z \in \operatorname{ob}\mathcal{C}$ a composition operation

$$\operatorname{Hom}_{\mathcal{C}}(y, z) \times \operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{C}}(x, z) \quad (g, f) \mapsto g \circ f$$

which is associative and has units $\operatorname{id}_x \in \operatorname{Hom}_{\mathcal{C}}(x, x)$.

When we will write $x \in \mathcal{C}$, we actually mean $x \in \operatorname{ob}\mathcal{C}$. Sometimes we will write $f\colon x \to y$ for $f \in \operatorname{Hom}_{\mathcal{C}}(x, y)$. We will also often omit the subscript $\mathcal{C}$ from the notation if it's clear what category we are talking about.

In a category, we often want to identify two objects not only when they are equal, but also when they are isomorphic:

Definition 3.7. We say that a morphism $f \in \operatorname{Hom}_{\mathcal{C}}(x, y)$ is an *isomorphism* if it has a both-sided inverse $f^{-1} \in \operatorname{Hom}_{\mathcal{C}}(y, x)$ under composition. Two objects are called *isomorphic* if there exists an isomorphism between them.

Example 3.8. The category of groups and group homomorphisms. Isomorphisms are group isomorphisms.

Example 3.9. The category pt is defined to have a single object and a single morphism.

Example 3.10. The category of Banach spaces and bounded operators. Isomorphisms are operators with a bounded inverse.

Another option would be to take morphisms to be norm-preserving. Then isomorphisms would be isometric isomorphisms. We will focus on the category of bounded operators in this document.

[2] We will see what Hausdorff spaces are later.

Example 3.11. The category of C^*-algebras and $*$-homomorphisms.

Example 3.12. The category of topological spaces and continuous maps. Isomorphisms are called *homeomorphisms.*

Example 3.13. If $\mathcal{C}$ is a category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects, but the composition is reversed. In other words, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(y,x)$ with composition in the reversed order.

So how do we express the relationship between the category of topological spaces and the category of C^*-algebras? First we need to know what is a 'morphism between categories'.

Definition 3.14. If $\mathcal{C}, \mathcal{D}$ are categories, a *functor* $\mathcal{C} \to \mathcal{D}$ consists of an assignment $F\colon \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ on objects, and a family of maps $F_{x,y}\colon \mathrm{Hom}(x,y) \to \mathrm{Hom}(Fx, Fy)$ for all objects $x, y \in \mathrm{ob}\,\mathcal{C}$ such that $F_{x,z}(g \circ f) = F_{y,z}(g) \circ F_{x,y}(f)$ and $F_{x,x}(\mathrm{id}_x) = \mathrm{id}_{Fx}$.

We usually abuse notation and write F for $F_{x,y}$.

Example 3.15. The assignment $X \mapsto C(X)$ defines a functor from the category of compact topological spaces to the opposite of the category of C^*-algebras. Indeed, you can verify that if $g\colon X \to Y$ is a continuous map, then $f \mapsto f \circ g$ defines a $*$-homomorphism $C(Y) \to C(X)$. Moreover, this respects identities and composition.

Example 3.16. If $\mathcal{C}$ is a category and $x \in \mathcal{C}$ is an object, there is a functor $\mathrm{pt} \to \mathcal{C}$ sending the unique object to x.

How do we say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ expresses that $\mathcal{C}$ are 'the same'? We could say that F has an inverse functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$. This notion is called *isomorphism of categories*, but we usually want something slightly weaker. For this, we will see that functors $\mathcal{C}_1 \to \mathcal{C}_2$ are themselves objects of a category. In that case, it might be better to talk about isomorphisms of functors as opposed to equality. Morphisms in the category of functors are called *natural transformations.*

Definition 3.17. If $F_1, F_2\colon \mathcal{C}_1 \to \mathcal{C}_2$ are functors, a natural transformation $F_1 \Rightarrow F_2$ is an assignment of $\phi_x\colon F_1(x) \to F_2(x)$ to $x \in \mathrm{ob}\,\mathcal{C}_1$ such that the diagram

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\phi_x} & F_2(x) \\ \big\downarrow{\scriptstyle F_1(f)} & & \big\downarrow{\scriptstyle F_2(f)} \\ F_1(y) & \xrightarrow{\phi_y} & F_2(y) \end{array}$$

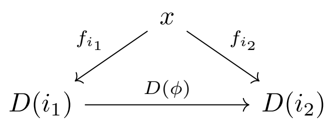
commutes for all $f \in \mathrm{Hom}_{\mathcal{C}_1}(x,y)$. Here a diagram *commutes* if it doesn't matter in which direction one traverses it. So the above diagram means $F_2(f) \circ \phi_x = \phi_y \circ F_1(f)$.

One can show that natural transformations assemble into a category with composition $(\psi \circ \phi)_x = \psi_x \circ \phi_x$. Now we can say that F_1 and F_2 are naturally isomorphic if they are isomorphic in this category. This is the same as saying that there exist a natural transformation $F_1 \Rightarrow F_2$ of which the components $\phi_x\colon F_1(x) \to F_2(x)$ are isomorphisms for all $x \in \mathcal{C}_1$.

Definition 3.18. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$.

A functor is an equivalence of categories if and only if it is 'fully faithful' and 'essentially surjective'. These conditions are good to know about as they are generally easier to check than to directly verify the existence of the inverse G, but we probably won't need them in this course.

Definition 3.19. Let $D\colon \mathcal{I} \to \mathcal{C}$ be a functor. An object $x \in \operatorname{ob}\mathcal{C}$ equipped with a collection of morphisms $f_i\colon x \to D(i)$ for all $i \in \operatorname{ob}\mathcal{I}$ is called a *limit* of D if it has *universal property* of being a *cone*: for all morphisms $\phi\colon i_1 \to i_2$ in $\mathcal{I}$, we have that

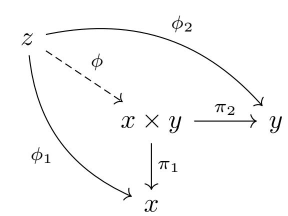


commutes[3] and for any other object $y \in \operatorname{ob}\mathcal{C}$ equipped with a collection of morphisms $g_i\colon y \to D(i)$ for all $i \in \operatorname{ob}\mathcal{I}$ such that the analogous diagram commutes, there exists a unique morphism $g\colon x \to y$ such that $f_i \circ g = g_i$ for all $i \in \operatorname{ob}\mathcal{I}$.

There is a completely analogous notion of *colimit* in which all the arrows are reversed.

Let I be a set and let $\mathcal{I} = \sqcup_{i\in I}\,\mathrm{pt}$ be the category with object set I, with only identity morphisms. Let $\mathcal{C}$ be a category. Then a functor $F\colon \mathcal{I} \to \mathcal{C}$ exactly picks out an I-family of objects $x_i \in \operatorname{ob}\mathcal{C}$ for $i \in I$. A *product* of the x_i, denoted $\prod_{i\in I} x_i$, is a limit of F.

For example, if $|I| = 2$, a product of x and y is an object $x \times y$ together with morphisms $\pi_1\colon x \times y \to x$ and $\pi_2\colon x \times y \to y$ such that for every pair of morphisms $\phi_1\colon z \to x$ and $\phi_2\colon z \to y$, there exists a unique morphism $\phi\colon z \to x \times y$ such that $\pi_1\phi = \phi_1$ and $\pi_2\phi = \phi_2$. This information is usually summarized by the commutative diagram:



Example 3.20. A product in the category of sets is given by the usual Cartesian product. A product in the category of groups is the usual product of groups.

Example 3.21. A product of topological spaces X_i in the category of topological spaces and continuous maps, is the product topology on $\prod_{i\in I} X_i$. This topology is generated by sets the form $\prod_{i\in I} U_i$, where $U_i \subseteq X$ are open and $U_i = X$ for all but finitely many $i \in I$.

Let $\mathcal{I}$ be the category consisting of two objects a and b and two morphisms $a \to b$ next to the two identity morphisms $a \to a$ and $b \to b$. Then a functor $F\colon \mathcal{I} \to \mathcal{C}$ is the same data as two objects $x, y \in \mathcal{C}$ and two morphisms $f, g\colon x \to y$. An *equalizer* of f and g is the limit of F.

Example 3.22. A equalizer of $f, g\colon G \to H$ in the category of groups is given by the kernel of fg^{-1}.

[3] Look at the diagram, it's a cone!

4 Series in Banach spaces and holomorphic functions (September 11)

In this lecture we will develop some analytic tools. This might be somewhat dry if you don't like analysis as much (like me), but it will be useful for understanding the spectrum of an element $a \in A$, and then the spectrum of a C^*-algebra A.

4.1 Convergent series

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in a Banach space X. We denote $\sum_{n=0}^{\infty} x_n := \lim_{k\to\infty} \sum_{n=0}^{k} x_n$ if the series converges.

Definition 4.1. We say that the series $\sum_{n\in\mathbb{N}} x_n$ *converges absolutely* if

$$\sum_n \|x_n\| < \infty.$$

Lemma 4.2. *If a series converges absolutely, then it converges.*

Proof. If x_n converges absolutely, then $(\sum_{n=0}^{N} x_n)_N$ is a Cauchy sequence because for $N_1 < N_2$ we have

$$\left\|\sum_{n=0}^{N_2} x_n - \sum_{n=0}^{N_1} x_n\right\| = \left\|\sum_{n=N_1}^{N_2} x_n\right\| \leq \sum_{n=N_1}^{N_2} \|x_n\|$$

and $(\sum_{n=0}^{N} \|x_n\|)_N$ is a Cauchy sequence since it converges. By completeness $(\sum_{n=0}^{N} x_n)_N$ converges. □

Let A be a Banach algebra. The fact that elements have a geometric series will be used often.

Lemma 4.3. *Let $a \in A$ satisfy $\|a\| < 1$. Then*

1. *$\sum_{n=0}^{\infty} a^n$ converges absolutely*
2. *$\sum_{n=0}^{\infty} a^n$ is the inverse of $1 - a$.*

Proof. By geometric series, we get

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = (1 - \|a\|)^{-1} < \infty.$$

Since $(\sum_{n=0}^{\infty} a^n)_N$ converges absolutely it converges to some element by Lemma 4.2. To verify it converges to $(1-a)^{-1}$, we take the limit of the equation

$$(1-a)(1 + a + a^2 + \cdots + a^N) = 1 - a^{N+1},$$

as $N \to \infty$, which by continuity of left multiplication by $1 - a$ gives us

$$(1-a)\left(\sum_n a^n\right) = 1,$$

since $\|a^N\| \leq \|a\|^N \to 0$ as $N \to \infty$. We can derive

$$\left(\sum_n a^n\right)(1-a) = 1$$

in a similar manner. □

Remark 4.4. For people who haven't seen geometric series: let $0 < r < 1$ be a real number. Note that

$$(1-r)(1+r+r^2+\cdots+r^N) = 1 - r^{N+1},$$

and so

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

for every N. Thus, since r^N converges to 0 (because $r < 1$), we see that

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

We need more generalities about convergence of series in Banach spaces. The following definitions and results are probably familiar in the special case that the Banach space is $\mathbb{C}$. For the next few lemmas, let x_n be a sequence in a Banach space, thought of as a series.

Definition 4.5. The *radius of convergence* of (x_n) is $R(x_n) := (\limsup_n \|x_n\|^{1/n})^{-1} \in [0, \infty]$.

Lemma 4.6. *Suppose that* $\sum_{n=0}^{\infty} x_n$ *converges. Then* $\lim x_n = 0$.

Proof. By convergence $\left(\sum_{n=0}^{k} x_n\right)_k$ is a Cauchy sequence. Let $\epsilon > 0$ and pick N large enough so that

$$\left\|\sum_{n=0}^{k} x_n - \sum_{n=0}^{l} x_n\right\| < \epsilon$$

if $k, l > N$. Now take $l = k+1$ to see that $\|x_n\| < \epsilon$ if $n > N$. □

Proposition 4.7 ([1, 13II])**.** *Let* $z \in \mathbb{C}$.

1. *If* $|z| < R(x_n)$*, then* $\sum_n x_n z^n$ *converges absolutely.*
2. *If* $\sum_n x_n z^n$ *converges, then* $z \leq R(x_n)$*.*

Proof. Suppose that $|z| < R(x_n)$. We must show that

$$\sum_m \|x_m\| \, |z|^m < \infty.$$

If $z = 0$, this is obvious, so assume that $|z| > 0$. Then since $R(x_n) > 0$, we have that $R(x_n)^{-1} |z| < 1$. We can therefore pick $\varepsilon > 0$ small enough so that $(R(x_n)^{-1} + \varepsilon) |z| < 1$. Now since $\limsup_m \|x_m\|^{\frac{1}{m}} <$

$R(x_n)^{-1}+\varepsilon$, we can take N so large that $\|x_m\|^{\frac{1}{m}} \leq R(x_n)^{-1}+\varepsilon$ for all $m \geq N$. Then $\|x_m\|^{\frac{1}{m}}\,|z| \leq (R(x_n)^{-1}+\varepsilon)\,|z| < 1$ for all $m \geq N$, and so

$$\begin{aligned}\sum_m \|x_m\|\,|z|^m &= \sum_{m=0}^{N-1} \|x_m\|\,|z|^m + \sum_{m=N}^{\infty} \left(\|x_m\|^{\frac{1}{m}}|z|\right)^m \\ &\leq \sum_{m=0}^{N-1} \|x_m\|\,|z|^m + \sum_{m=N}^{\infty} (\,(R(x_n)^{-1}+\varepsilon)\,|z|\,)^m < \infty\end{aligned}$$

by convergence of the geometric series.

Suppose now instead that $\sum_n x_n z^n$ converges. Then $\|x_n\||z|^n$ converges to 0 by Lemma 4.6. In particular, there is N with $\|x_m\|\,|z|^m \leq 1$ for all $m \geq N$. Then $\|x_m\|^{\frac{1}{m}} \leq |z|^{-1}$ for all $m \geq N$, so that $R(x_n)^{-1} = \limsup_m \|x_m\|^{\frac{1}{m}} \leq |z|^{-1}$, giving $|z| \leq R(x_n)$. ☐

4.2 Integration

We will define integrals in Banach spaces. There will be no surprises, but it is also somewhat tedious. For this we need a lemma which is also useful in many other contexts. To define the terms in the lemma, we make a brief topological detour:

Definition 4.8. A subset Y of a topological space X is called *dense* if $\overline{Y} = X$. Here we used the notation $\overline{A}$ for the *closure* of a subset $A \subseteq X$ of a topological space X; the intersection of all closed sets in X which contain A.

Exercise 5. Show that an arbitrary intersection of closed sets in a topological space X is closed. Conclude that the closure of a subset $A \subseteq X$ is closed.

Lemma 4.9. *Let $D \subseteq X$ be a dense subspace of a normed space X and let Y be a Banach space. If $T\colon D \to Y$ is a bounded linear map, then T extends uniquely to an operator $T'\colon X \to Y$ with the same operator norm.*

Proof. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D converging to x. If T' exists, we must have

$$T'x = \lim_{x_n \to x} Tx_n.$$

Since x_n converges in X, x_n is a Cauchy sequence in D. Because T is bounded, it follows that Tx_n is a Cauchy sequence and so this limit exists.

Note first that T' is well-defined. Indeed, if x'_n is another sequence converging to x, then $x_n - x'_n$ converges to 0, so that $\|Tx_n - Tx'_n\|$ converges to 0. From this it follows that

$$\lim_{x_n\to x} Tx_n = \lim_{x'_n \to x} Tx'_n$$

as desired.

We show that T' is bounded with the same norm as[4] as follows

$$\|T'x\| = \|\lim_{n\to\infty} T(x_n)\| = \lim_{n\to\infty} \|T(x_n)\| \leq \lim_{n\to\infty} \|T\|\|x_n\| = \|T\|\|x\|$$

☐

[4]If x_n converges to x, then $\|x\| = \lim_{n\to\infty}\|x_n\|$ by the reverse triangle inequality. In other words, $\|.\|\colon X \to \mathbb{R}$ is a continuous map.

Let X be a Banach space and take a half open[5] interval $(s, t]$ in $\mathbb{R}$. Define a *step function* $f\colon (s,t] \to X$ to be a linear combination of *indicator functions* of subintervals. Here a subinterval is a subset $(s_1, t_1] \subseteq (s,t]$ for some $s \leq s_1 < t_1 \leq t$ and the indicator function on a set $S \subseteq (s,t]$ with value $x \in X$ is the function

$$f(t') = \begin{cases} x & \text{if } t' \in S, \\ 0 & \text{if } t' \notin S. \end{cases}$$

If f is a step function, it is possible to pick $s = s_0 < s_1 < \cdots < s_k < s_{k+1} = t$ for some $k \geq 0$ so that $f(t')$ takes a fixed value $x_i \in X$ whenever $s_i < t' \leq s_{i+1}$ for all $i \in \{0, \ldots, k\}$. Moreover, given f there is a unique minimal choice of s_j (in the sense that k is smallest) where we have that $x_i \neq x_{i+1}$ for all i. The integral of this step function is then defined as

$$\int_s^t f = \sum_{i=0}^{k} x_i(s_{i+1} - s_i) \in X.$$

Every step function has a canonical extension to $[s,t]$ and from now on we will consider them as functions on $[s,t]$. Step functions form a linear subspace $S[s,t] \subseteq \ell^\infty([s,t];X)$, where $\ell^\infty([s,t];X)$ is the Banach space of bounded maps $[s,t] \to X$ with the supremum norm $\sup_{t' \in [s,t]} \|f(t')\|$.

The integral defines a linear map $\int_s^t \colon S(s,t] \to X$ which is bounded because

$$\left\| \int_s^t f(t') dt' \right\| \leq (t-s)\|f\|.$$

It therefore extends to a bounded linear map $\int_s^t \colon \overline{S[s,t]} \to X$ by Lemma 4.9. Let $C([s,t];X) \subseteq \ell^\infty([s,t];X)$ the subspace of continuous functions $[s,t] \to X$. Step functions are rarely continuous, but it turns out that we do have $C([s,t];X) \subseteq \overline{S}$:

Exercise 6. Show that every continuous function $[s,t] \to X$ is a uniform limit of step functions. (Hint: by the Heine–Cantor theorem, such a function is uniformly continuous)

Let $\Gamma \subseteq \mathbb{C}$ be a *contour*, i.e. a subset that can be parametrized by a C^1 path $\gamma\colon [s,t] \to \mathbb{C}$. The *contour integral* of a continuous map $f\colon C \to X$ is defined by

$$\int_\Gamma f = \int_s^t f(\gamma(t'))\gamma'(t')dt'$$

One can check that this is independent of the choice of parametrization using the chain rule and the fact that any reparameterization corresponds to a C^1 map $\tau\colon [s_1, t_1] \to [s_2, t_2]$.

4.3 Holomorphic functions

Functions which are complex differentiable are usually called holomorphic functions. Holomorphic functions from the complex numbers into a Banach space make sense. Even more, we will see later that all cool complex analysis theorems such as Cauchy's integral theorem still hold.

Definition 4.10. Let X be a Banach space. If $U \subseteq \mathbb{C}$ is an open set, a function $f\colon U \to X$ is *holomorphic* if for all $z \in U$

$$f'(z) := \lim_{w \to z} \frac{f(z) - f(w)}{z - w}$$

exists.

[5]We do this for technical reasons. It is also possible to allow intervals of all four types.

Exercise 7.

1. Given $x \in X$, show that the map $\mathbb{C} \to X$ given by $z \mapsto zx$ is holomorphic with derivative the constant function at x.
2. Show that if $U, V \subseteq \mathbb{C}$ are open, $f\colon V \to X$ is holomorphic and $g\colon U \to V$ is holomorphic in the usual complex analysis sense, then $f \circ g$ is holomorphic.
3. Show that if A is a Banach algebra and $f\colon U \to A$ and $g\colon U \to A$ are holomorphic, then $f \cdot g\colon U \to A$ is holomorphic.

References

[1] Abraham (Bram) Westerbaan. The category of von Neumann algebras. *arXiv preprint arXiv:1804.02203*, 2018.